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Stability of Impulsive Differential Equation with any Time Delay

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ABSTRACT: In this paper, the stability of general impulsive retarded functional differential equations with any time delay has been considered. Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. Impulsive differential equations, that is, differential equations involving impulse effects, are a natural description of observed evolution phenomena of several real world problems. Impulsive control which based on impulsive differential equations has attracted the interest of many researchers recently. The method of Lyapunov functions and Razumikhin technique have been widely applied to stability analysis of various delay differential equation. When Lyapunov functions are used, it becomes necessary to choose an appropriate minimal class of functionals relative to which the derivative of the Lyapunov function is estimated. This approach is known as the Lyapunov–Razumikhin technique. When Lyapunov functionals are used the corresponding derivative can be estimated without demanding minimal classes of functional. By using Lyapunov functions and analysis technique along with Razumikhin technique, some results for the uniform stability of such impulsive differential equations have been derived. The obtained results extend and generalize some results existing in the literature.

KEYWORDS: Impulsive delay systems, Lyapunov function, Razumikhin technique, Uniform stability, Time delays.

1 INTRODUCTION

A Impulsive differential equations have attracted many researchers' attention due to their wide applications in many fields such as control technology, drug administration and threshold theory in biology, industrial robotics, sociology, chemistry and so on. Many classical results have been extended to impulsive systems [1], [10], [11]. By Lyapunov's direct method, various stability problems have been discussed for impulsive delay differential equations[9] .On the other hand, there have been many papers and monographs recently on stability analysis of delay differential equations [4], [8]. The method of Lyapunov functions and Razumikhin technique have been widely applied to stability analysis of various delay differential equations, and they have also proved to be a powerful tool in the investigation of asymptotical properties of impulsive delay differential equations [3]. There are several research works appeared in the literature on impulsive delayed linear differential equations. In this paper we will study the stability of impulsive differential equation for any time delays. As a result criteria on uniform stability can be derived.

This paper is organized as follows. In Section II, we introduce some basic definitions and notations. In Section III, we get some criteria for stability of impulsive differential equations with any time delay. Finally, concluding remarks are given in Section IV.

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2 PRELIMINARIES

Consider the following impulsive differential:

$$\begin{cases} x'(t) = f(t, x_t), t \neq t_k, t \geq t_0 \\ \Delta x(t_k) = I_k\left(t_k, x_{t_k^-}\right), k \in N \\ x_{t_0} = \eta \end{cases} \quad \text{(i)}$$

We assume that function f, $I_k: R_+ \times PC([-\tau, 0], R^n) \to R^n)$ and $\eta \in PC([-\tau, 0], R^n)$ satisfy all required conditions for existence and uniqueness of the solutions for all $t \geq t_0$.

The time sequence $[t_k]_{k=1}^{+\infty}$ satisfies 0=$t_0 < t_1 < \cdots < t_k < \cdots, \lim_{k\to 0} t_k = \infty$,

$\Delta x(t_k) = x(t_k) - x(t_k^-)$ and $x_t, x_{t^-} \in PC([-\tau, 0], R^n)$ are defined by $x_t(r) = x(t + r)$ and

$x_{t^-}(r) = x(\mathrm{t}^- + \mathrm{r})$ for $-\tau \leq r \leq 0$ respectively. We shall assume that f(t,0)= $I_k(t, 0) = 0$ for all t ∈ R_+ and k ∈ N, so that system (i) admits the trivial solution.

Given a constant $\tau > 0$, we equip the linear space $PC([-\tau, 0], R^n)$ with the norm $\|.\|_\tau = sup_{-\tau\leq r\leq 0}\|\psi(s)\|$. Denote x(t)=x(t,σ,η), for σ≥ $t_0$be the solution of (i) s.t.

$x(\sigma + r) = \eta(s), s\epsilon[-\tau, 0]$.

We further assume that all the solutions x(t) of (i) are continuous except at t_k,k ∈ N at which x(t) is right continuous.

Definition 1: The function *V* : *R*+ × R^n → *R*+ is said to belong to the class ν_0 if we have the following.

1) V is continuous in each of the sets $[t_{k-1}, t_k) \times R^n$, and for each x ∈ R^n,t ∈ $[t_{k-1}, t_k)$ k ∈ N,

$lim_{(t,w)\to(t_k^-,x)} V(t, w) = V(t_k^-, x)$ exists.

2) *V* (*t, x*) is locally Lipschitzian in all *x* ∈ R^n,, and for all *t* ≥ t_0, *V* (*t,* 0) ≡ 0.

Definition 2: Given a function *V* : *R*+ × R^n → *R*+, the upper right-hand derivative of *V* with respect to system (i) is defined by $D^+V(t, x(t)) = \lim\sup_{\delta\to 0^+} \frac{1}{\delta}[V(t+\delta, x(t+\delta)) - V(t, x(t))]$.

3 MAIN RESULTS

In the following, we shall establish criteria on impulsive differential equation with any time delay for uniform stability. We have the followings results.

Theorem 1: Assume that there exist a function V ∈ $\nu_0$and some positive constants *p,c,*c_1, c_2, c_3, c_4, c_5 such that 0<c_5<1.

(a) $c_1\|x\|^p \leq V(t, x) \leq c_2\|x\|^p$, for any t ∈ R^+ and x ∈ R^n

(b) $D^+V(t, \psi(0)) \leq cV(t, \psi(0))$, for all t ∈ $[t_{k-1}, t_k)$ k ∈ N

Whenever s $V(t, \psi(0)) \geq V(t + r, \psi(r))$ for r ∈ $[-\tau, 0]$, where $s \geq \alpha e^{\beta\tau}$ is a constant, where $\alpha \geq 1, \beta > 0$.

(c) There exist a positive constant ξ_k, where $0 < \xi_{k-1} \leq 1, \forall k \in N$

$$s.t. V\left(t_{k,} \psi(0) + I\left(t_{k,}, \psi\right)\right) \leq \xi_k V(t_k^-, \psi(0)), \frac{1}{\alpha} \leq \xi_{k-1}.$$

(d) $-(c_3 + \frac{c_4}{c_5})(t_k - t_{k-1}) > \ln c_5, , \forall k \in N$ the zero solution of (i) is uniformly stable.

Proof: Let x(t)=x(t, t_0, η) be any solution of the impulsive system (i) with the initial condition $x_{t_0} = \eta$ and *v*(t)=V(t,x).

We have:

$c_1\|x\|^p \leq v(t) \leq c_2\|x\|^p, \ \ when\ t \neq \tau_k, k = 1,2,3 \ldots$ (ii)

$v'(t) \leq \ c_3 v(t) + c_4 v(t-\tau)$, for any σ≥ t_0, η∈ $PC(\Omega)$

Let x(t)=x(t,σ,η) be the solution of (i) through (σ, η).Let σ ∈ $[\tau_{m-1}, \tau_m)$ for some m ∈ N.

First of all we will prove that

$v(t) \leq \frac{c_2}{c_5}\Omega^2$ (iii)

Where $\Omega^2 < \left(\frac{c_1 c_5}{c_2}\right)\gamma^p,$ for any $\gamma > 0$

Observably, we know that for any t ∈ ([$\sigma - \tau, \sigma$], there a a ∈ $[-\tau, 0]$ such that t= σ+a

So, we have

$v(t) = v(\ \sigma + a) \leq \ c_2\|x\|^p \leq c_2\Omega^2 < \frac{c_2}{c_5}\Omega^2, \sigma - \tau \leq t < \ \sigma$

So if inequality (iii) does not hold, then there is a r' ∈ $(\ \sigma, \tau_m)$ such that

$v(r') > \frac{c_2}{c_5}\Omega^2 > c_2\Omega^2 > v(\ \sigma\ )$

Then from the continuity of *v*(t) in [σ, τ_m), there exist a r_1 ∈ $(\ \sigma, r')$ such that

$v(r_1) = \frac{c_2}{c_5}\Omega^2$

v(t)≤ $\frac{c_2}{c_5}\Omega^2, \sigma - \tau \leq t < r_1$

$v'(r_1) \geq 0$ *(*iv*)*

From inequalities (iv), $\frac{c_2}{c_5}\Omega^2 > c_2\Omega^2, v(t) \leq c_2\Omega^2$

For t ∈ ([$\sigma - \tau, \sigma$], it follows that there exist r_2 ∈ $[\ \sigma, r_1)$ such that $v(r_2) = c_2\Omega^2$

v(t)≥ $c_2\Omega^2, r_2 \leq t \leq r_1$

$v'(r_2) \geq 0$ (v)

From the inequalities (iv), *v(t)*≤ $\frac{c_2}{c_5}\Omega^2, \sigma - \tau \leq t < r_1$

From the inequalities (v), *v*(t)≥ $c_2\Omega^2$,t∈ $[r_2, r_1]$

∴ for t∈ $[r_2, r_1]$

$v(t + r) \leq \frac{c_2}{c_5}\Omega^2 \leq \frac{1}{c_5}\Omega^2, r \in [-\tau, o]$

So, $v(t-\tau) \leq \frac{1}{c_5}\Omega^2$, t∈ $[r_2, r_1]$

$v'(t) \leq \left(c_3 + \frac{c_4}{c_5}\right) v(t)$

$\frac{v'(t)}{v(t)} \leq \left(c_3 + \frac{c_4}{c_5}\right)$

$\int_{r_2}^{r_1} \frac{v'(t)}{v(t)} dt \leq \int_{r_2}^{r_1} \left(c_3 + \frac{c_4}{c_5}\right) dt \leq \int_{\tau_{m-1}}^{\tau_m} \left(c_3 + \frac{c_4}{c_5}\right) dt = \left(c_3 + \frac{c_4}{c_5}\right)(\tau_m - \tau_{m-1}) < -ln\, c_5$

At the same time

$\int_{r_2}^{r_1} \frac{v'(t)}{v(t)} dt = \int_{v(r_2)}^{v(r_1)} \frac{dy}{y} = \int_{c_2\Omega^2}^{\frac{c_2}{c_5}\Omega^2} \frac{dy}{y} = ln\left(\frac{c_2}{c_5}\Omega^2\right) - ln(c_2\Omega^2) = -ln\, c_5$

Which is a contradiction, so (iii) holds

From inequality (iii) & given conditions, we obtain

$v(\tau_m) \leq c_5 v(\tau_m^-) \leq c_2\Omega^2$ (vi)

Next, we prove that

$v(t) \leq \frac{c_2}{c_5}\Omega^2, \tau_m \leq t \leq \tau_{m+1}$ (vii)

If this does not hold,then there is $\hat{u} \in (\tau_m, \tau_{m+1})$

Such that $v(\hat{u}) > \frac{c_2}{c_5}\Omega^2 > c_2\Omega^2 \geq v(\tau_m)$

From the continuity of V(t,x(t)) in (τ_m, τ_{m+1}), there exist a $u_1 \in (\tau_m, \hat{u})$ such that

$v(u_1) = \frac{c_2}{c_5}\Omega^2$

$v(t) \leq \frac{c_2}{c_5}\Omega^2, \sigma - \tau \leq t < u_1$

$v'(u_1) \geq 0$ (viii)

From the inequalities $\frac{c_2}{c_5}\Omega^2 > c_2\Omega^2$, $v(\tau_m) \leq c_2\Omega^2$

It follows that there exist a $u_2 \in (\tau_m, u_1)$ such that $v(u_2) = \frac{c_2}{c_5}\Omega^2$

$v(t) \geq c_2\Omega^2, u_2 \leq t \leq u_1$

Therefore, for $t \in [u_2, u_1]$

$v(t+r) \leq \frac{c_2}{c_5}\Omega^2 \leq \frac{1}{c_5} v(t)$

So $v(t-\tau) \leq \frac{1}{c_5} v(t)$, then for $t \in [u_2, u_1]$

$v'(t) \leq \left(c_3 + \frac{c_4}{c_5}\right) v(t)$

$\frac{v'(t)}{v(t)} \leq \left(c_3 + \frac{c_4}{c_5}\right)$ (ix)

So, integrating (ix) in $t \in [u_2, u_1]$, we get

$\int_{u_2}^{u_1} \frac{v'(t)}{v(t)} dt \leq \int_{u_2}^{u_1} \left(c_3 + \frac{c_4}{c_5}\right) dt \leq \int_{\tau_m}^{\tau_{m+1}} \left(c_3 + \frac{c_4}{c_5}\right) dt = \left(c_3 + \frac{c_4}{c_5}\right)(\tau_{m+1} - \tau_m) < -ln\, c_5$

At the same time

$\int_{u_2}^{u_1} \frac{v'(t)}{v(t)} dt = \int_{v(u_2)}^{v(u_1)} \frac{dy}{y} = \int_{c_2\Omega^2}^{\frac{c_2}{c_5}\Omega^2} \frac{dy}{y} = ln\left(\frac{c_2}{c_5}\Omega^2\right) - ln(c_2\Omega^2) = -ln\, c_5$

Which is a contradiction. So (vii) holds.

From inequality (vii) and given conditions, we have

$v(\tau_{m+1}) \leq c_5 v(\tau_{m+1}^-) \leq c_2\Omega^2$

By mathematical induction, we can prove that in general for k=0,1,2,…

$v(t) \leq c_2\Omega^2$

$v(t) \leq \frac{c_2}{c_5}\Omega^2, \tau_{m+k} \leq t \leq \tau_{m+k+1}$

$v(\tau_{m+k+1}) \leq c_2\Omega^2$

which together with inequality (iii) and the inequality $\frac{c_2}{c_5}\Omega^2 \geq c_2\Omega^2$

provides $v(t) \leq \frac{c_2}{c_5}\Omega^2, t \geq \sigma$

So, $c_1\|x\|^p \leq v(t) \leq \frac{c_2}{c_5}\Omega^2, t \geq \sigma$

$\|x\|^p \leq \frac{c_2}{c_1 c_5}\Omega^2$

$\|x\| \leq (\frac{c_2}{c_1 c_5})^{\frac{1}{p}}\Omega^{\frac{2}{p}} < \gamma, t \geq \sigma$

Therefore, the zero solution of (i) is uniformly stable

Theorem 2: If c_3, c_5, c_6 be any constants and $c_7 = max(c_5, c_6)$ then zero solution of (i) is uniformly stable

if $c_3(\tau_k - \tau_{k-1}) < -\ln c_7$, for $k \in N$.

Proof: Let $c_1>0$, $c_2>0, c_4>0$ be any constants. Let us consider that for $\gamma > 0$, there exist $\Omega > 0$ such that

Where $\Omega^2 < \left(\frac{c_1 c_7}{c_2 + \tau c_4}\right)\gamma^p$

Construct a Lyapnov function $v(t) \in v_0$ and let

$c_1\|x(t)\|^p \leq v(t) \leq c_2\|x(t)\|^p + \tau c_4 sup_{-\tau \leq r \leq 0}\|x(t+r)\|^p$

When $t \neq \tau_k$, k=1,2,… we have $v'(t) \leq c_3 v(t)$

For any $\sigma \geq t_0$ and $\eta \in PC(\Omega)$, let x(t)=x(t,σ,η) be the solution of (i) through (σ,η). Let $\sigma \in [\tau_{m-1}, \tau_m)$ for some $m \in N$.We first prove that

$v(t) \leq \frac{c_2 + \tau c_4}{c_7}\Omega^2, \sigma \leq t \leq \tau_m$ (x)

When t= σ, $v(t) = v(\sigma) \leq (c_2 + \tau c_4)\eta^2 \leq (c_2 + \tau c_4)\Omega^2 \leq \frac{c_2 + \tau c_4)}{c_7}\Omega^2$

So if inequality (x) does not hold, then there is $\hat{r} \in (\sigma, \tau_m)$ such that

$$v(\hat{r}) > \frac{c_2+\tau c_4)}{c_7}\Omega^2 > (c_2 + \tau c_4)\Omega^2 \geq v(\sigma)$$

Then from the continuity of *v*(t) in $[\sigma, \tau_m)$, there exist a $r_1 \in (\sigma, \hat{r})$ such that

$$v(r_1) = \frac{c_2+\tau c_4)}{c_7}\Omega^2$$

$$v(t) = \frac{c_2+\tau c_4)}{c_7}\Omega^2, \sigma \leq t \leq r_1$$

$$v'(r_1) \geq 0$$

From the inequalities $\frac{c_2+\tau c_4)}{c_7}\Omega^2 > (c_2 + \tau c_4)\Omega^2$, $v(\sigma) \leq (c_2 + \tau c_4)\Omega^2$, it follows that there exist a $r_2 \in (\sigma, r_1)$, such that: $v(r_2) = (c_2 + \tau c_4)\Omega^2$

$$v(t) \geq (c_2 + \tau c_4)\Omega^2, r_2 \leq t \leq r_1$$

$$v'(r_2) \geq 0$$

Therefore integrate the inequality $v'(t) \leq c_3 v(t)$ in $t \in [r_2, r_1]$, we have

$$\int_{r_2}^{r_1} \frac{v'(t)}{v(t)} dt \leq \int_{r_2}^{r_1} c_3 dt \leq \int_{\tau_{m-1}}^{\tau_m} c_3 dt = c_3(\tau_m - \tau_{m-1}) < -ln\ c_7$$

At the same time :

$$\int_{r_2}^{r_1} \frac{v'(t)}{v(t)} dt = \int_{v(r_2)}^{v(r_1)} \frac{du}{u} = \int_{(c_2+\tau c_4)\Omega^2}^{\frac{c_2+\tau c_4)}{c_7}\Omega^2} \frac{du}{u} = \ln\left(\frac{c_2+\tau c_4)}{c_7}\Omega^2\right) - \ln\left((c_2 + \tau c_4)\Omega^2\right) = -\ln\ c_7$$

A contradiction, So (x) holds and from the given conditions we have

$$v(\tau_m) \leq c_7 v(\tau_m^-) \leq (c_2 + \tau c_4)\Omega^2$$

Similar to the proof before, we can easily get the following inequality

$$v(t) \leq \frac{c_2+\tau c_4)}{c_7}\Omega^2, \tau_m \leq t \leq \tau_{m+1}$$

By mathematical induction, we can prove that in general for k=0,1,2,…

$$v(t) \leq \frac{c_2+\tau c_4)}{c_7}\Omega^2, \tau_{m+k} \leq t \leq \tau_{m+k+1}$$

$$v(\tau_{m+k+1}) \leq (c_2 + \tau c_4)\Omega^2$$

Which together with equation (x) and the inequality $\frac{c_2+\tau c_4)}{c_7}\Omega^2 \geq (c_2 + \tau c_4)\Omega^2$ provides

$$v(t) \leq \frac{c_2+\tau c_4)}{c_7}\Omega^2, t \geq \sigma$$

So $c_1\|x(t)\|^p \leq v(t) \leq \frac{c_2+\tau c_4)}{c_7}\Omega^2, t \geq \sigma$

$$\|x\| \leq \left(\frac{c_2+\tau c_4}{c_7 c_1}\right)^{\frac{1}{p}} \Omega^{\frac{2}{p}} < \gamma, t \geq \sigma$$

Therefore the zero solution of (i) is uniformaly stable and the proof of the Theorem is complete.

4 CONCLUSION

In this paper, we have considered the stability of impulsive differential equations with any time delay. By using Lyapnov functions and techniques along with Razumikhin technique, we have gotten some results for the uniform stability of impulsive differential equations with any time delay.

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