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Optimization of Economic Order Quantity and Profits under Markovian Demand

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ABSTRACT: Most manufacturers, wholesalers and retailers face a situation of stock depletion over time. Replenishment is usually made using the Economic Order Quantity (EOQ) model. The model assumes deterministic demand of a single item; often at a constant rate whose total inventory costs (ordering and holding) per unit time are minimized. In this paper, a new approach is developed to optimize the economic order quantity (EOQ) of a single item, finite horizon, and periodic review inventory problem with stochastic demand at optimum profits. In the given model, sales price and inventory replenishment periods are uniformly fixed over the planning horizon. Adopting a Markov decision process approach, the states of a Markov chain represent possible sates of demand for the inventory item. The ordering cost, holding cost, shortage cost and sales price are combined with demand and inventory positions to generate profits for the EOQ decision problem. The objective is to determine in each period of the planning horizon an optimal economic order quantity so that the long run profits are maximized for a given state of demand. Using weekly equal intervals, the decisions of how much to order are made using dynamic programming over a finite period planning horizon. A numerical example demonstrates the existence of an optimal state-dependent economic order quantity as well as the corresponding profits of item.

KEYWORDS: Dynamic programming, Inventory, Markov chains, Ordering, States.

1 INTRODUCTION

Business enterprises continuously strive to plan for optimal inventory levels that sustain random demand of items. In practice, when inventory exceeds quantity demanded, inventory carrying costs accumulate which affect profit margins of the enterprise. Similarly, inventory levels below demand impose shortage costs and loss of good will from potential customers. Both cases drastically reduce profit margins unless proper planning and coordination is put in place to establish optimal inventory levels in a given business enterprise. In an effort to achieve this goal, two major problems are usually encountered:

(i) Determining the economic order quantity (EOQ) of the item in question

(ii) Determining the optimal profits associated with the economic Order quantity (EOQ) when demand is uncertain. In this paper, an inventory system is considered whose goal is to optimize the economic Order quantity (EOQ) and profits associated with ordering and holding inventory of an item. At the beginning of each period, a major has to be made: namely whether to order additional units of the item in inventory or cancel ordering decisions and utilize the available units in inventory. In either case, the economic Order quantity (EOQ) must be determined that optimize profits in a given inventory system.

According to Hung [1], the EOQ can be examined under stochastic demand and discounting when backlogging is present or not. Results indicate that without discounting, the stochastic variability of demand has no impact on EOQ.With discounting however, the EOQ rises along with the demand variance parameter. In a related article by Eyan[2],the problem of determining effective and simple EOQ-like solutions for stochastic demand is examined. The article considers a periodic review system under stochastic demand with variable stock outs. Halim[3] develops an EOQ model in fuzzy sense by considering stochastic demand, partial backlogging and fuzzy deterioration rate.

The three models presented have some interesting insights regarding economic order quantity in terms of optimization and stochastic demand. However, profit maximization is not a salient factor in the optimization models cited.

2 Model Formulation

2.1 Notation and Assumptions

i,j = States of demand

D^K_{ij} = Quantity demanded

U = Unfavorable state

n,N = Stages

K= Ordering policy

N^K = Customer matrix

D^k = Demand matrix

I^K = Inventory matrix

P^K = Profit matrix

Q^K=Demand transition matrix

p_s = Unit sales price

N^K_{ij} = Number of customers

F = Favorable state

I^K_{ij} = Quantity in inventory

P^K_{ij} = Profits

V^K_i = Expected profits

a^K_i = Accumulated profits

c_o = Unit ordering cost

c_h = Unit holding cost

c_s = Unit shortage cost

Q^K_{ij}= Demand transition probability

O^K_i = Economic Order Quantity

$i,j \in \{F,U\}$ $\quad K \in \{0,1\}$ $\quad n=1,2,\ldots\ldots\ldots\ldots N$

We consider an inventory system of a single item whose demand during a chosen period over a fixed planning horizon is classified as either *Favorable* (state F) or *Unfavorable* (state U) and the demand of any such period is assumed to depend on the demand of the preceding period. The transition probabilities over the planning horizon from one demand state to another may be described by means on a Markov chain. Suppose one is interested in determining the optimal course of action, namely to order additional units of item (a decision denoted by K=1) or not to order additional units (a decision denoted by K=0) during each time period over the planning horizon where K is a binary decision variable. Optimality is defined such that the maximum expected profits are accumulated at the end of N consecutive time periods spanning the planning horizon. In this paper, a two-period (N=2) planning horizon is considered.

2.2 Finite dynamic programming formulation

Recalling that the demand can be in state F or in state U, the problem of finding an optimal economic order quantity may be expressed as a finite period dynamic programming model.

Let $g_n(i)$ denote the expected total profits accumulated during periods *n,n+1,........,N* given that the state of the system at the beginning of period *n* is $i\varepsilon\{F,U\}$.The recursive equation relation g_n and g_{n+1} is

$$g_n(i) = max_K[Q^K_{iF}(P^K_{iF} + g_{n+1}(F)), Q^K_{iU}(P^K_{iU} + g_{n+1}(U))] \quad (1)$$

$i\varepsilon\{F,U\}$,*n=1,2,................................N* together with the final conditions $g_{N+1}(F) = g_{N+1}(U) = 0$

This recursive relationship may be justified by noting that the cumulative total profits $P^K_{ij}+ g_{n+1}(j)$ resulting from reaching $j\varepsilon\{F,U\}$at the start of period *n+1* from state $i\varepsilon\{F,U\}$at the start of period *n* occurs with probability Q^K_{ij}.

Clearly, $V^K = Q^K (P^K)^T$, $K\varepsilon \{0,1\}$ (2)

where "T" denotes matrix transposition, and hence the dynamic programming recursive equations

$$g_n(i) = max_K[V^K + Q^K_{iF}g_{n+1}(F) + Q^K_{iU}g_{n+1}(U)] \quad (3)$$

$i\varepsilon\{F,U\}$ $n=1,2,\ldots\ldots\ldots N-1$, $K\varepsilon\{0,1\}$.

$$g_N(i) = max_K[V_i^K] \quad i\varepsilon\{F,U\} \quad (4)$$

result where (4) represents the Markov chain stable state.

3 COMPUTING Q^K, P^K AND O^K

The demand transition probability from state i ε{F, U}to state jε{F, U } given ordering policy Kε {0,1} may be taken as the number of customers observed when demand is initially in state i and later with demand changing to state j, divided by the number of customers over all states.

$$Q_{ij}^K = N_{ij}^K/[N_{iF}^K + N_{iU}^K] \quad (5)$$

iε{F,U} , K ε {0,1}

When demand outweighs on-hand inventory, the profit matrix P^Z may be computed by the relation:

$$P^K = p_s[D^K] - (c_0 + c_h + c_s)(D^K - I^K)$$

Therefore

$$P_{ij}^K = \begin{cases} p_s D_{ij}^K - (c_0 + c_h + c_s)(D_{ij}^K - I_{ij}^K) & if \quad D_{ij}^K > I_{ij}^K \\ p_s D_{ij}^K \quad if \quad D_{ij}^K \leq I_{ij}^K \end{cases} \quad (6)$$

for all i,j ε{F,U}, K ε {0,1}

A justification for expression (6) is that $D^K_{ij} - I^K_{ij}$ units must be ordered in order to meet the excess demand. Otherwise ordering is cancelled when demand is less than or equal to on-hand inventory. The economic order quantity when demand is initially in state i ε{F,U},given ordering policy Kε {0,1} is

$$O_i^K = (D_{iF}^K - I_{iF}^K) + (D_{iU}^K - I_{iU}^K) \quad (7)$$

iε{F,U}, Kε {0,1}

The following conditions must however hold:

1.$O^K_i > 0$ when $D^K_{ij} > I^K_{ij}$ and $O^K_{ij} = 0$ when $D^K_{ij} \leq I^K_{ij}$

2.K=1 when $c_0 > 0$, and K=0 when $c_0 = 0$

3.$c_s > 0$ when shortages are allowed and $c_s = 0$ when shortages are not allowed.

4 COMPUTING AN OPTIMAL ECONOMIC ORDER QUANTITY

The optimal EOQ is found in this section for each time period separately,

4.1 OPTIMIZATION DURING PERIOD 1

When demand is Favorable (i.e. in state F), the optimal ordering policy during period 1 is

$$K = \begin{cases} 1 & if \quad v_F^1 > v_F^0 \\ 0 & if \quad v_F^1 \leq v_F^0 \end{cases}$$

The associated total profits and EOQ are then

$$g_1(F) = \begin{cases} v_F^1 & if \quad K = 1 \\ v_F^0 & if \quad K = 0 \end{cases}$$

and

$$O_F^K = \begin{cases} (D_{FF}^1 - I_{FF}^1) + (D_{FU}^1 - I_{FU}^1) & if \quad K = 1 \\ 0 \quad if \quad K = 0 \end{cases}$$

respectively. Similarly, when demand is Unfavorable (i.e. in state U), the optimal ordering policy during period 1 is

$$K = \begin{cases} 1 & if \quad v_U^1 > v_U^0 \\ 0 & if \quad v_U^1 \leq v_U^0 \end{cases}$$

In this case, the associated total profits and EOQ are

$$g_1(U) = \begin{cases} v_U^1 & if \quad K = 1 \\ v_U^0 & if \quad K = 0 \end{cases}$$

and

$$O_U^K = \begin{cases} (D_{UF}^1 - I_{UF}^1) + (D_{UU}^1 - I_{UU}^1) & if \quad K = 1 \\ 0 \quad if \quad K = 0 \end{cases}$$

respectively.

4.2 OPTIMIZATION DURING PERIOD 2

Using dynamic programming recursive equation (1), and recalling that a_i^Z denotes the already accumulated profits at the end of period 1 as a result of decisions made during that period,it follows that:

$$a_i^K = v_i^K + Q_{iF}^K max[v_F^1, v_F^0] + Q_{iU}^K max[v_U^1, v_U^0]$$
$$= v_i^K + Q_{iF}^K \, g_1(F) + Q_{iU}^K \, g_1(U)$$

Therefore when demand is favorable (i.e. in state F), the optimal ordering policy during period 2 is

$$K = \begin{cases} 1 & if \quad a_U^1 > a_U^0 \\ 0 & if \quad a_U^1 \leq a_U^0 \end{cases}$$

while the associated total profits and EOQ are

$$g_2(F) = \begin{cases} a_F^1 & if \quad K = 1 \\ a_F^0 & if \quad K = 0 \end{cases}$$

and

$$O_F^K = \begin{cases} (D_{FF}^1 - I_{FF}^1) + (D_{FU}^1 - I_{FU}^1) & if \quad K = 1 \\ 0 \quad if \quad K = 0 \end{cases}$$

Similarly, when demand is Unfavorable (i.e. in state U), the optimal ordering policy during period 2 is

$$K = \begin{cases} 1 & if \quad a_U^1 > a_U^0 \\ 0 & if \quad a_U^1 \leq a_U^0 \end{cases}$$

while the associated total profits and EOQ are

$$g_2(U) = \begin{cases} a_U^1 & if \quad K = 1 \\ a_U^0 & if \quad K = 0 \end{cases}$$

and

$$O_U^K = \begin{cases} (D_{UF}^1 - I_{UF}^1) + (D_{UU}^1 - I_{UU}^1) & if \quad K = 1 \\ 0 \quad if \quad K = 0 \end{cases}$$

5 CASE STUDY

In order to demonstrate the use of the model in §2-4, a real case application from *Steel and Tube Hardware* , a hardware in Uganda is presented in this section. The demand for Iron sheets fluctuates from month to month. The company wants to avoid ordering when demand is low or not ordering when demand is high, and hence seeks decision support in terms of an optimal ordering policy , the associated profits and specifically, a recommendation as to the EOQ of iron sheets over a two-week period.

5.1 DATA COLLECTION

A sample of 60 customers was used. Past data revealed the following demand pattern and inventory levels over the state transitions for twelve weeks in Tables 1 and 2 below:

Table 1. Customers, Demand and Inventory at state-transitions over twelve weeks for Ordering Policy 1(K = 1)

State Transition (i,j)	Customers N^1_{ij}	Demand D^1_{ij}	Inventory I^1_{ij}
FF	40	80	74
FU	20	20	60
UF	10	120	60
UU	50	40	10

Table 2. Customers, Demand and Inventory at state-transitions over twelve weeks for Ordering Policy 0(K=0)

State Transition (i,j)	Customers N^0_{ij}	Demand D^0_{ij}	Inventory I^0_{ij}
FF	30	50	20
FU	30	30	40
UF	20	160	80
UU	40	80	20

When additional iron sheets are ordered K=1) during week 1,

$$N^1 = \begin{bmatrix} 40 & 20 \\ 10 & 50 \end{bmatrix} \qquad D^1 = \begin{bmatrix} 80 & 20 \\ 120 & 40 \end{bmatrix} \qquad I^1 = \begin{bmatrix} 74 & 60 \\ 60 & 10 \end{bmatrix}$$

while if additional iron sheets are *not* ordered during week 1, these matrices are

$$N^0 = \begin{bmatrix} 30 & 30 \\ 20 & 40 \end{bmatrix} \qquad D^0 = \begin{bmatrix} 50 & 30 \\ 160 & 80 \end{bmatrix} \qquad I^0 = \begin{bmatrix} 20 & 40 \\ 80 & 20 \end{bmatrix}$$

In either case, the unit sales price(p_s) is \$20.00,the unit ordering cost (c_0) is \$15.00,the unit holding cost per week(c_h) is \$0.50,and the unit shortage cost per week(c_s) is \$10.00

5.2 COMPUTATION OF MODEL PARAMETERS

Using (5) and (6), the state transition matrix and the profit matrix (in thousand dollars) for week 1 are

$$Q^1 = \begin{bmatrix} 0.67 & 0.33 \\ 0.17 & 0.83 \end{bmatrix} \qquad P^1 = \begin{bmatrix} 1.447 & 1.570 \\ 0.870 & 0.035 \end{bmatrix}$$

for the case where additional iron sheets are ordered during week 1, while these matrices are given by

$$Q^0 = \begin{bmatrix} 0.50 & 0.50 \\ 0.33 & 0.67 \end{bmatrix} \qquad P^0 = \begin{bmatrix} 0.235 & 0.580 \\ 1.160 & 0.070 \end{bmatrix}$$

for the case when additional iron sheets are *not* ordered during week 1.

When additional iron sheets are ordered (K=1), the matrices Q^1 and P^1 yield the profits (in thousand dollars)

$$v_F^1 = (0.67)(1.447) + (0.33)(1.570) = 1.488$$

$$v_U^1 = (0.17)(0.870) + (0.83)(0.035) = 0.177$$

However, when additional iron sheets are not ordered (K=0), the matrices Q^0 and P^0 yield the profits (in US dollars)

$$v_F^0 = (0.50)(0.235) + (0.50)(0.580) = 0.408$$

$$v_U^0 = (0.33)(1.160) + (0.67)(0.07) = 0.430$$

5.3 THE OPTIMAL ORDERING POLICY AND EOQ

Since 1.488 > 0.408, it follows that K=1 is an optimal Ordering policy for week 1 with associated total profits of $1.488 and an EOQ of 80-74 = 6 units when demand is favorable. Since 0.430 > 0.177, it follows that K = 0 is an optimal Ordering policy for week 1 with associated total profits of $0.430 and an EOQ of 0 units when demand is unfavorable.

If demand is favorable, then the accumulated profits at the end of week 1 are

$$a_F^1 = 1.488 + (0.67)(1.488) + (0.33)(0.430) = 2.447$$

$$a_F^0 = 0.408 + (0.50)(1.488) + (0.50)(0.430) = 1.367$$

Since 2.447 > 1.367, it follows that K = 1 is an optimal Ordering policy for week 1 with associated accumulated profits of $ 2.447 and an EOQ of 80 -74 = 6 units for the case of favorable demand.

However, if demand is unfavorable, then the accumulated profits at the end of week 1 are

$$a_U^1 = 0.177 + (0.17)(1.488) + (0.83)(0.430) = 0.786$$

$$a_U^0 = 0.430 + (0.33)(1.488) + (0.67)(0.430) = 1.209$$

Since 1.209 > 0.786, it follows that K=0 is an optimal Ordering policy for week 2 with associated accumulated profits of $1.209 and an EOQ of 0 units for the case of unfavorable demand.

When shortages are not allowed, the values of K and $g_n(i)$ and O^K_i may be computed for i ε{F, U}in a similar fashion after substituting $c_s = 0$ the matrix function

$$P^K = p_s[D^K] - (c_0 + c_h + c_s)[D^K - I^K]$$

6 CONCLUSION

An inventory model was presented in this paper. The model determines an optimal ordering policy, profits and the EOQ of a given item with stochastic demand. The decision of whether or not to order additional inventory units is modeled as a multi-period decision problem using dynamic programming over a finite period planning horizon. The working of the model was demonstrated by means of a real case study.

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