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A positivity-preserving nonstandard finite difference scheme for a system of reaction-diffusion equations with nonlocal initial conditions

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ABSTRACT: A significant number of ecological phenomena can be modeled using nonlinear reaction-diffusion partial differential equations. This paper considers a system of reaction-diffusion equations with nonlocal initial conditions. Such equations arise as steady-state equations in an age-structured predator-prey model with diffusion. We use the non-standard finite difference method developed by Mickens, which is a scheme that preserves the positivity of solutions. Furthermore, this scheme is explicit and functional relationship is obtained between the time, the space, and age step sizes.

KEYWORDS: Predator-prey model, nonlinear diffusion, nonlocal initial conditions, finite difference methods, nonstandard finite difference schemes, positivity of solutions.

1 INTRODUCTION

The application of partial differential equations is very common in the natural and engineering sciences. In particular, the nonlinear modeling of reaction and/or diffusion phenomena [1]. In this paper we consider the equations for studying an age-structure of the predator-prey model, defined by a parabolic equations with nonlocal initial conditions, which in one dimension of space take the form [2] :

$$\begin{cases} \partial_t u + \partial_a u = d_1 \partial_{xx} u - (\alpha_1 u + \alpha_2 v) u \,, t \geq 0, a \in (0, a_m), x \in \Omega \\ \partial_t v + \partial_a v = d_2 \partial_{xx} v - (\beta_1 v - \beta_2 u) v \,, t \geq 0, a \in (0, a_m), x \in \Omega \end{cases} \quad (1)$$

Where $u = u(t, a, x) \geq 0$ and $v = v(t, a, x) \geq 0$ are subject to the constraints

$$\begin{cases} u(t, 0, x) = \int_0^{a_m} \eta b_1(a) u(t, a, x) da, t \geq 0, x \in \Omega \\ v(t, 0, x) = \int_0^{a_m} \xi b_2(a) v(t, a, x) da, t \geq 0, x \in \Omega \end{cases} \quad (2)$$

Here u and v are respectively the densities of populations of predator and prey that live in the same spatial region Ω and are structured by age $a \in (0, a_m)$ and with spatial position $x \in \Omega$ space.

The parameter $a_m > 0$ is the maximum age of two species, while constants $d_1, d_2 > 0$ give the rates at which the species diffuse. For notational simplicity, we will take $d_1 = d_2 = 1$.

The mortality rates in the equations in (1) are given by

$$\begin{cases} \mu_1(u, v) = \alpha_1 u + \alpha_2 v \\ \mu_2(u, v) = \beta_1 v - \beta_2 u \end{cases} \quad (3)$$

The equations in (1) represent the age limits conditions and reflect that individuals who have age zero are created when a single generator of any age $a \in (0, a_m)$ gives birth with rates $\eta b_1(a)$ and $\xi b_2(a)$ respectively. The functions $b_j = b_j(a) > 0$ describe the profiles the fertility rates while the parameters $\eta, \xi > 0$ measure their intensities without affecting the structure of the birth rate.

The objective of this paper is to construct a finite difference scheme that preserves the positivity of solutions. The method used in this construction is based on the nonstandard discretization technique created by Ronald Mickens. In particular, the scheme must be explicit and requires a functional relationship between time, space, and age step sizes. In general, the absence of this restriction leads to the existence of numerical instabilities, i.e., there exits solutions to the finite difference equations that do not correspond qualitatively to any of the possible solution to the differential equations [3].

The remainder of the paper is organized as follows: Section 2 gives a brief outline of the nonstandard finite difference scheme (NSFDS) philosophy, while Section 3 presents the construction of a new NSFDS. Finally, Section 4 gives a summary discussion of the results obtained.

2 NONSTANDARD MODELING

The procedures for constructing NSFDS are stated in [3]. These types of schemes are made by the following two rules [4],[5]: first, how to determine the discrete representations of derivatives, and secondly, what is the form of nonlinear terms.

The first concept involves the generalization of the representations of derivatives form [2],[3]:

$$\frac{dx}{dt} \to \frac{x_{n+1} - x_n}{\phi(h, \lambda)} \tag{4}$$

Where $t_n = (\Delta t)n = nh, x_n$ is the approximation of $x(t_n)$, and the denominator function satisfies :

$$\phi(h, \lambda) = h + \vartheta(h^2) \tag{5}$$

In equations (4) and (5), λ represents the parameters of the differential equation. This construction technique can easily be extended to partial differential equations.

The second concept is related to the modeling of nonlinear terms, for example, x^2.

In general, the nonlinear terms are replaced by nonlocal representation [4]:

$$x^2 \to \begin{cases} 2x_n^2 - x_{n+1}x_n \\ x_{n+1}x_n \end{cases} \tag{6}$$

Both concepts are discussed in detail in [3]. A good example is the advection-reaction equation [4],[6]

$$u_t + u_x = u(1 - u) \tag{7}$$

Where the exact numerical scheme is

$$\frac{u_m^{n+1} - u_m^n}{\phi(\Delta t)} + \frac{u_m^n - u_{m-1}^n}{\phi(\Delta x)} = u_{m-1}^n(1 - u_m^{n+1}) \tag{8}$$

The denominator function is

$$\phi(z) = e^z - 1 \tag{9}$$

$t_n = (\Delta t)n, x_m = (\Delta t)m$ and u_m^n is the representation of $u(x_m, t_n)$. The major difficulty in the numerical simulation of partial differential equations of which we do not know the exact solution often leads to consider

$$\phi(z) = z \tag{10}$$

3 NONSTANDARD FINITE DIFFERENCE SCHEME

A possible nonstandard finite difference scheme for system (1) is

$$\begin{cases} \dfrac{u_{m,p}^{n+1}-u_{m,p}^{n}}{\Delta t}+\dfrac{u_{m,p}^{n}-u_{m-1,p}^{n}}{\Delta a}=d_1\dfrac{u_{m,p+1}^{n}-2u_{m,p}^{n}+u_{m,p-1}^{n}}{(\Delta x)^2} \\ \qquad -\alpha_1 u_{m,p}^{n+1}u_{m-1,p}^{n}-\alpha_2 u_{m,p}^{n+1}v_{m-1,p}^{n} \\ \dfrac{v_{m,p}^{n+1}-v_{m,p}^{n}}{\Delta t}+\dfrac{v_{m,p}^{n}-v_{m-1,p}^{n}}{\Delta a}=d_2\dfrac{v_{m,p+1}^{n}-2v_{m,p}^{n}+v_{m,p-1}^{n}}{(\Delta x)^2} \\ \qquad -\beta_1 v_{m,p}^{n+1}v_{m-1,p}^{n}+\beta_2 u_{m-1,p}^{n}v_{m-1,p}^{n} \end{cases} \tag{11}$$

Where we have the following discrete representations:

$$\begin{cases} \partial_t u \to \dfrac{u_{m,p}^{n+1}-u_{m,p}^{n}}{\Delta t}\ ;\ \partial_a u \to \dfrac{u_{m,p}^{n}-u_{m-1,p}^{n}}{\Delta a} \\ \partial_{xx} u \to \dfrac{u_{m,p+1}^{n}-2u_{m,p}^{n}+u_{m,p-1}^{n}}{(\Delta x)^2}\ ;\ \partial_{xx} v \to \dfrac{v_{m,p+1}^{n}-2v_{m,p}^{n}+v_{m,p-1}^{n}}{(\Delta x)^2} \\ -u^2 \to -u_{m,p}^{n+1}\cdot u_{m-1,p}^{n}\ ;\ -uv \to -u_{m,p}^{n+1}\cdot v_{m-1,p}^{n}\ ;\ uv \to u_{m-1,p}^{n}\cdot v_{m-1,p}^{n} \end{cases} \tag{12}$$

Solving $u_{m,p}^{n+1}$ and $v_{m,p}^{n+1}$ in (11), we get

$$\begin{cases} u_{m,p}^{n+1}=\dfrac{u_{m,p}^{n}(1-R_1-2d_1R_2)+R_1u_{m-1,p}^{n}+d_1R_2\left(u_{m,p+1}^{n}+u_{m,p-1}^{n}\right)}{1+\alpha_1\Delta t u_{m-1,p}^{n}+\alpha_2\Delta t v_{m-1,p}^{n}} \\ v_{m,p}^{n+1}=\dfrac{v_{m,p}^{n}(1-R_1-2d_2R_2)+R_1v_{m-1,p}^{n}+d_2R_2\left(v_{m,p+1}^{n}+v_{m,p-1}^{n}\right)+\beta_2\Delta t u_{m-1,p}^{n}v_{m-1,p}^{n}}{1+\beta_1\Delta t v_{m-1,p}^{n}} \end{cases} \tag{13}$$

Where

$R_1=\frac{\Delta t}{\Delta a}$ and $R_2=\frac{\Delta t}{(\Delta x)^2}$ (14)

Positivity of solutions provides that

$$u_{m,p}^{n}\ge 0\,,\,v_{m,p}^{n}\ge 0 \Rightarrow u_{m,p}^{n+1}\ge 0, v_{m,p}^{n+1}\ge 0 \tag{15}$$

Which leads to the conditions

$$\begin{cases} 1-R_1-2d_1R_2\ge 0 \\ 1-R_1-2d_2R_2\ge 0 \end{cases} \tag{16}$$

And gives

$$\begin{cases} \Delta t \le \dfrac{(\Delta a)(\Delta x)^2}{(\Delta x)^2+2d_1(\Delta a)} \\ \Delta t \le \dfrac{(\Delta a)(\Delta x)^2}{(\Delta x)^2+2d_2(\Delta a)} \end{cases} \tag{17}$$

Taking $d_1=d_2=1$, we find that

$$\Delta t \le \frac{(\Delta a)(\Delta x)^2}{(\Delta x)^2+2(\Delta a)} \tag{18}$$

In this paper, we only consider the equality in equation (18) and system (13) becomes :

$$\begin{cases} u_{m,p}^{n+1}=\dfrac{R_1u_{m-1,p}^{n}+R_2\left(u_{m,p+1}^{n}+u_{m,p-1}^{n}\right)}{1+\alpha_1\Delta t u_{m-1,p}^{n}+\alpha_2\Delta t v_{m-1,p}^{n}} \\ v_{m,p}^{n+1}=\dfrac{R_1v_{m-1,p}^{n}+R_2\left(v_{m,p+1}^{n}+v_{m,p-1}^{n}\right)+\beta_2\Delta t u_{m-1,p}^{n}v_{m-1,p}^{n}}{1+\beta_1\Delta t v_{m-1,p}^{n}} \end{cases} \tag{19}$$

Where $R_1=\frac{(\Delta x)^2}{(\Delta x)^2+2(\Delta a)}$, $R_2=\frac{(\Delta a)}{(\Delta x)^2+2(\Delta a)}$ and $2R_2+R_1=1$

It is clear that the solutions of the system (19) are positive. For simulation, select $\Delta a, \Delta x$ and then calculate Δt , R_1 and R_2.

4 CONCLUSION

A nonstandard finite difference scheme was constructed for the system of parabolic equations which describes an age-structure predator-prey model with nonlocal initial conditions. The most important aspect of this scheme is that it satisfies the positivity condition. We are currently attempting to generalize this work to the parabolic systems with cross-diffusion equation and nonlocal initial conditions.

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