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# Analytical solutions of Lame equations by Galerkin method

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**ABSTRACT:** This paper deals with the deformation of an elastic solid described by Lame equations satisfying the boundary conditions. By means of the differential operators, we reduce these equations to Poisson equations that we solve using Galerkin method, i.e. we obtain the components of displacement vector. Furthermore, we compute the strain and stress tensors acting on the solid which are important in engineering applications. Numerous examples are given in this work.

**KEYWORDS:** Deformation of the solid, Strain and stress tensors, Boundary condition, Poisson equations, Galerkin method

## 1 INTRODUCTION

The theory of elasticity studies the small deformation of a solid which occupies a domain bounded by a surface under various boundary conditions. For describing the physical phenomena in theory of elasticity one uses the mathematical model of solid deformation such as Lame equations. It is a well-known fact that a few exact solutions of the Lame equations are known even now. This has been largely due to the complexity of the system of differential equations. In the absence of general solution, it is often convenient to experiment with models to obtain information on the deformation phenomena e.g. the displacement vector, deformation and stress tensors, etc. Lame equations constitute a system of partial differential equations. The study of these equations is motivated by their role in many applications in various fields of physics and technology. In many remarkable papers [1-11] the solids and structures are studied intensively.

0ne of the methods for studying partial differential equations is group analysis. This analytical approach based on symmetries of differential equations was originally introduced by Sophus Lie and further developed by [12-16]. For each system of partial differential equations there is symmetry group, that acts on the space of its independent and dependent variables, leaving the form of the system unchanged.

The mixed finite element methods have been proposed by Arnold [17] and Qiu [18] for solving linear elasticity problems. Bustinza [19] and Yuncheng [20] have developed a general framework of constructing discontinuous Galerkin methods for solving the linear elasticity problem. Assous [21] used the Nitsche method for studying the Navier-Lame equations.

In the present work, and for the first time to our best knowledge of the literature, we extend the continuous Galerkin method to vectorial Lame equations.

The paper is organized as follows. In the section 2, we present the theoretical framework describing the deformation of an elastic solid. In the third section, using the differential operators we reduce Lame equations to Poisson equations. Then applying Galerkin method to the latter equations subjected to boundary conditions, we establish the integral relations which allow to obtain the solutions of Lame equations, i.e. the displacement vector. Furthermore, we compute the strain and stress tensors acting on the solid. In the fourth section, numerous examples are given. In the last section, the concluding remarks follow.

## 2 STATEMENT OF THE PROBLEM

Consider an elastic homogenous body occupying a three-dimensional domain $\Omega$ bounded by a closed differentiable surface $S$.

The deformation of a solid is governed by the Lame equations Eglit[22] and Parton[23]:

$$\text{«}\Delta A + (\text{ª} + \text{«})\nabla(\nabla.A) + \text{°}F = 0 \ \text{ in } \Omega \tag{1}$$

$$A = A_0 \text{ on } S \tag{2}$$

where $A = (u, v, w) \in \Re^3$, $F = (P, Q, R) \in \Re^3$ are the displacement vector and the body force respectively at any point $x = (x_1, x_2, x_3) \in \Omega$; $A_0 = (u_0, v_0, w_0)$ is the constant displacement vector imposed on $S$; «, ª are Lame coefficients of the solid and ° is the solid density.

Let $\text{£}_{i,j}$ and $\pm_{i,j}$, $i, j = 1,2,3$ be the components of the strain and stress tensors respectively defined by Eglit[22] and Parton[23] as:

$$\text{£}_{i,j} = \frac{1}{2}\left[\frac{\partial b_i}{\partial x_j} + \frac{\partial b_j}{\partial x_i}\right],\ \pm_{i,j} = \text{ª¢}_{ij}\nabla.A + 2\text{«£}_{ij} \tag{3}$$

where $b_1 = u$, $b_2 = v$ and $b_3 = w$ stands for Kronecker symbol.

## 3 ANALYTICAL SOLUTION

In this section we provide analytical solutions to the problem (1),(2).

For this purpose, the system (1) can be written in the form

$$\Delta u + (1 + \text{ª}/\text{«})\frac{\partial}{\partial x_1}(\nabla.A) + (\text{°}/\text{«})P = 0 \ \text{ in } \Omega \tag{4}$$

$$\Delta v + (1 + \text{ª}/\text{«})\frac{\partial}{\partial x_2}(\nabla.A) + (\text{°}/\text{«})Q = 0 \ \text{ in } \Omega \tag{5}$$

$$\Delta w + (1 + \text{ª}/\text{«})\frac{\partial}{\partial x_3}(\nabla.A) + (\text{°}/\text{«})R = 0 \ \text{ in } \Omega \tag{6}$$

Applying the operators $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ to the equations (4),(5), (6) respectively and

summing the results, we reduce the previous system to Poisson equation

$$\Delta(\nabla.A) = \frac{-\text{°}}{2\text{ª} + \text{«}}\nabla.F \ \text{ in } \Omega \tag{7}$$

subject to the following boundary condition

$$\nabla.A = 0 \text{ on } S \tag{8}$$

In order to solve the problem (7) and (8), we consider the following problem

$$\Delta\text{¤} = -f(x) \ \text{ in } \Omega \tag{9}$$

$$\text{¤} = \text{¤}_0(x) \ \text{ on } S \tag{10}$$

The following statement holds

**Theorem 1** (*Vladimirov [24])*

Assume that $f \in C(\overline{\Omega})$, $¤_0 \in C(S)$. If the function ¤ has normal derivative to $S$, a solution ¤ of the problem (9), (10) is

$$¤(x) = -\int_S \frac{\partial G(x,y)}{\partial n_y} ¤_0(y) dS_y + \int_\Omega G(x,y)\, f(y) dy \tag{11}$$

where $n_y$ stands for the outward normal vector to $S$ at the point $y \in S$ and $G$ is Green function of Laplacian operator.

**Remark**

Let us point out that one can find a solution ¤ through the n-th order linearly independent harmonic homogenous polynomials, spherical functions, Bessel functions, etc [25].

By virtue of the formula (11), the equations (4)-(6) satisfying the boundary condition (2) become

$$-\Delta u = \frac{°(« + ª)}{«(2ª + «)} \int_\Omega \frac{\partial G(x,y)}{\partial x_1} \nabla.F(y) dy + (°/«)P \tag{12}$$

$$u|_S = u_0 \tag{13}$$

$$-\Delta v = \frac{°(« + ª)}{«(2ª + «)} \int_\Omega \frac{\partial G(x,y)}{\partial x_2} \nabla.F(y) dy + (°/«)Q \tag{14}$$

$$v|_S = v_0 \tag{15}$$

$$-\Delta w = \frac{°(« + ª)}{«(2ª + «)} \int_\Omega \frac{\partial G(x,y)}{\partial x_3} \nabla.F(y) dy + (°/«)R \tag{16}$$

$$w|_S = w_0 \tag{17}$$

Applying the Galerkin method to the problems (12) and (13), (14) and (15), (16) and (17) we obtain the solutions

$$u(x) = u_0 + ¨(x),\ v(x) = v_0 + ·(x),\ w(x) = w_0 + ¤(x)$$

where ¨ , · , ¤ are twice continuously differentiable functions defined on $\Omega \cup S$ satisfying the following integral relations

$$\int_\Omega (\nabla ¨(x))^2 dx = \frac{°}{«(2ª + «)} \int_\Omega \left[ (« + ª) \int_\Omega \frac{\partial G(x,y)}{\partial x_1} \nabla.F(y) dy + (2ª + «) P(x) \right] ¨(x) dx \tag{18}$$

$$\int_\Omega (\nabla ·(x))^2 dx = \frac{°}{«(2ª + «)} \int_\Omega \left[ (« + ª) \int_\Omega \frac{\partial G(x,y)}{\partial x_2} \nabla.F(y) dy + (2ª + «) Q(x) \right] ·(x) dx \tag{19}$$

$$\int_\Omega (\nabla ¤(x))^2 dx = \frac{°}{«(2ª + «)} \int_\Omega \left[ (« + ª) \int_\Omega \frac{\partial G(x,y)}{\partial x_3} \nabla.F(y) dy + (2ª + «) R(x) \right] ¤(x) dx \tag{20}$$

Here we formulate the main result:

**Theorem 2**

The displacement vector components $u$ , $v$ , $w$ of the solid deformation described by Lame equations (1) satisfying the boundary conditions (2) and the components of the strain and stress tensors £$_{ij}$ , ±$_{ij}$ , $i$ , $j = 1, 2, 3$ are defined by

$$u(x) = u_0 + ¨(x),\ v(x) = v_0 + ·(x),\ w(x) = w_0 + ¤(x) \tag{21}$$

$$£_{11} = \frac{\partial ¨}{\partial x_1}(x),\ £_{12} = \frac{1}{2}\left[\frac{\partial ¨}{\partial x_2}(x) + \frac{\partial ·}{\partial x_1}(x)\right],\ £_{12} = £_{21}\ ; \tag{22}$$

$$£_{13} = \frac{1}{2}\left[\frac{\partial ¨}{\partial x_3}(x) + \frac{\partial ¤}{\partial x_1}(x)\right],\ £_{13} = £_{31}\ ; \tag{23}$$

$$£_{22} = \frac{\partial ·}{\partial x_2}(x),\ £_{23} = \frac{1}{2}\left[\frac{\partial ·}{\partial x_3}(x) + \frac{\partial ¤}{\partial x_2}(x)\right]\ ; \tag{24}$$

$$£_{33} = \frac{\partial ¤}{\partial x_3}(x),\ £_{32} = £_{23}\ ; \tag{25}$$

$$±_{11} = \frac{°ª}{2ª + «}\int_\Omega G(x, y)\nabla.F(y)dy + 2«£_{11},\ ±_{12} = 2«£_{12}\ ,\ ±_{12} = ±_{21}\ ; \tag{26}$$

$$±_{22} = \frac{°ª}{2ª + «}\int_\Omega G(x, y)\nabla.F(y)dy + 2«£_{22},\ ±_{23} = 2«£_{23}\ ,\ ±_{23} = ±_{32}\ ; \tag{27}$$

$$±_{33} = \frac{°ª}{2ª + «}\int_\Omega G(x, y)\nabla.F(y)dy + 2«£_{33},\ ±_{31} = 2«£_{13}\ ,\ ±_{31} = ±_{13}\ ; \tag{28}$$

where ¨ , · , ¤ are defined from the integral relations (18),(19), (20).

## 4 APPLICATIONS

In order to compute strain and stress tensors acting on the solid in the engineering applications, we define a solid $\Omega$ bounded by a closed surface $S$ and the part $S_1 \subset S$ by:

$$\Omega = \{x \in R^3 : -~ < x_1 < ~; -\quad < x_2 < \quad ; 0 < x_3 < x_1^2 + x_2^2\};$$

$$S_1 = \{x \in R^3 : -~ \le x_1 \le ~; -\quad \le x_2 \le \quad ; x_3 = 0\};\ ~ > 0,\quad > 0\,.$$

Obviously, the projection on the $x_1\, x_2$ plane of a solid $\Omega$ represents a circular section bounded by a circle in which is inscribed the rectangle $S_1$.

### 4.1 EXAMPLE 1

We consider

$$\Delta(\nabla.A) = -e^{-x_3}\sin x_1 \cos x_2 \quad \text{in} \quad \Omega\,; \tag{29}$$

$$\nabla.A = 0\,,\ x_3 = 0\,. \tag{30}$$

We can easily see that

$$\nabla.A = \left(e^{-x_3} - e^{-x_3\sqrt{2}}\right)\sin x_1 \cos x_2 \tag{31}$$

is the exact solution of the problem.

### 4.2 Example 2

Find the solution of the following problem

$$\Delta(\nabla.A) = 0 \text{ in } \Omega\,; \tag{32}$$

$$\nabla.A = \text{¯}\left(x_2 - x_1\right),\ x_3 = 0 \tag{33}$$

where ¯ is Heavyside function.

We can write

$$\nabla.A = \frac{1}{2} + \frac{1}{\text{®}}\arctan\left(\frac{x_2 - x_1}{x_3\sqrt{2}}\right) \tag{34}$$

as the exact solution of the problem.

### 4.3 Displacement vector, strain and stress tensors

We consider the constant body force $F = (P_0, Q_0, R_0)$, i.e. $\nabla.F = 0$. Without limiting the generality, we choose

$$\nabla.A = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_3^2;\ x \in \Omega$$

as a solution of the equation (7) and we impose that $\nabla.A$ takes the value 0 on $S_1$

Set

$$\text{¨}(x) = a\left(\sim^2 - x_1^2\right)\left(\ ^2 - x_2^2\right)x_3,\ \cdot(x) = b\left(\sim^2 - x_1^2\right)\left(\ ^2 - x_2^2\right)x_3,\ \text{¤}(x) = c\left(\sim^2 - x_1^2\right)\left(\ ^2 - x_2^2\right)x_3$$

satisfying

$$\text{¨}(\mathrm{x})|_{S1} = 0,\ \cdot(\mathrm{x})|_{S1} = 0,\ \text{¤}(\mathrm{x})|_{S1} = 0$$

where $a, b, c$ are constants to determine.

By virtue of the formulas (18),( 19) and (20) derived from Galerkin method, the displacement vector components $u, v, w$ and the components of the strain and stress tensors $\text{£}_{ij}$, $\pm_{ij}$, $i, j = 1, 2, 3$ can be expressed as :

$$u(x) = u_0 + a\left(\sim^2 - x_1^2\right)\left(\ ^2 - x_2^2\right)x_3,\ v(x) = v_0 + b\left(\sim^2 - x_1^2\right)\left(\ ^2 - x_2^2\right)x_3\ ,$$

$$w(x) = w_0 + c\left(\sim^2 - x_1^2\right)\left(\ ^2 - x_2^2\right)x_3;$$

$$\text{£}_{11} = -2ax_1\left(\ ^2 - x_2^2\right)x_3,\ \text{£}_{12} = -a\left(\sim^2 - x_1^2\right)x_2x_3 - bx_1\left(\ ^2 - x_2^2\right)x_2$$

$$\text{£}_{13} = \frac{1}{2}a\left(\sim^2 - x_1^2\right)\left(\ ^2 - x_2^2\right) - cx_1\left(\ ^2 - x_2^2\right)x_3,\ ;\quad \text{£}_{12} = \text{£}_{21}\ ,\ \text{£}_{13} = \text{£}_{31}\ ;$$

$$\text{£}_{22} = -2b\left(\sim^2 - x_1^2\right)x_2\,x_3,\ \text{£}_{23} = \frac{1}{2}b\left(\sim^2 - x_1^2\right)\left(\ ^2 - x_2^2\right) - c\left(\sim^2 - x_1^2\right)x_2x_3;$$

$$\text{£}_{33} = c\left(\sim^2 - x_1^2\right)\left(\ ^2 - x_2^2\right),\ \text{£}_{32} = \text{£}_{23}$$

$$\pm_{11} = \text{ª}\left(x_1 + x_2 - 2x_3\right) + 2\text{«£}_{11},\ \pm_{12} = 2\text{«£}_{12}\ ,\ \pm_{13} = 2\text{«£}_{13}\ ;$$

$$\pm_{22} = ª\left(x_1 + x_2 - 2x_3\right) + 2«£_{22}, \pm_{21} = \pm_{12}\ , \pm_{23} = 2«£_{23}\ ;$$

$$\pm_{33} = ª\left(x_1 + x_2 - 2x_3\right) + 2«£_{33}, \pm_{31} = \pm_{13}\ , \pm_{23} = \pm_{23}\ ;$$

where $a$ , $b$ , $c$ are defined by:

$$a = \frac{¢_1}{¥}\ ,\quad b = \frac{¢_2}{¥}\ ,\quad c = \frac{¢_3}{¥} \tag{35}$$

$$¢_1 = \frac{8°}{«} P_0 \left( \frac{1}{105} \sim^{7}\ {}^{3} + \frac{2}{225} \sim^{5}\ {}^{5} + \frac{1}{105} \sim^{3}\ {}^{7} \right)$$

$$¢_2 = \frac{8°}{«} Q_0 \left( \frac{1}{105} \sim^{7}\ {}^{3} + \frac{2}{225} \sim^{5}\ {}^{5} + \frac{1}{105} \sim^{3}\ {}^{7} \right)$$

$$¢_3 = \frac{8°}{«} R_0 \left( \frac{1}{105} \sim^{7}\ {}^{3} + \frac{2}{225} \sim^{5}\ {}^{5} + \frac{1}{105} \sim^{3}\ {}^{7} \right)$$

$$¥ = \frac{128}{3} \left( \frac{1}{135} \left( \sim^{9}\ {}^{5} + \sim^{5}\ {}^{9} \right) + \frac{2}{245} \sim^{7}\ {}^{7} + \frac{1}{525} \left( \sim^{5}\ {}^{9} + \sim^{9}\ {}^{5} \right) + \frac{1}{2079} \left( \sim^{3}\ {}^{11} + \sim^{11}\ {}^{3} \right) \right) + \frac{256}{1575} \left( \sim^{7} + \sim^{5}\ {}^{2} \right)$$

## 5 CONCLUSION

We have investigated Lame equations that we reduced to Poisson equations using the differential operators. Applying Galerkin method to the latter equations we obtained the displacement vector components. Doing so, we computed the components of the strain and stress tensors. Examples have been given in this work proving the efficiency of Galerkin method.

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