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Cauchy-Kowaleskya problem in fuzzy normed spaces

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ABSTRACT: This paper deals with abstract version of the Cauchy Problem in fuzzy normed space. We define a Hausdorff measure of non compactness for bounded fuzzy set to prove existence of solutions by using a sequential approximation of the abstract problem. As a byproduct, we obtain a fuzzy version of the Cauchy-Kowaleskya Theorem for the generalized Hukuhara nonlinear partial differential equations.

KEYWORDS: Fuzzy normed spaces, generalized Hukuhara derivative, Hausdorff measure of non compacteness.

MATH. SUBJECT CLASSIFICATION: 35R13, 34A07, 28E10.

1 INTRODUCTION

This paper deals with some abstract version of fuzzy Cauchy - Kowaleskya Problem in (abstract) fuzzy normed space for the existence of solution to the problem:

$$u'_{gH} = A(t, u) \quad t \in I \subset \mathbb{R},\ I =]0, a_0[,\ a_0 > 0 \qquad (1)$$
$$u(0) = u_0.$$

In crisp and also in fuzzy cases (1) has been studied by many authors see for instance [3], [4] and [5] for the crisp case and [6], [7] and [9] for the fuzzy case. Although, for the fuzzy case, the authors consider the nonlinear function A continuous in both of its variables. This paper is concerned with the case where A could be only continuous with respect to u and measurable with respect to t. The lack of compactness of the most of fuzzy normed or metric spaces could be a big deal for the existence of solution to the problem (1), and therefore a big challenge that should be solved. For this purpose, it is natural to consider (1) under some non-compactness hypothesis. In crisp case, such studies have been done by some authors, see for instance [4] and references therein.

In this paper, we define a Hausdorff measure of non-compactness for bounded fuzzy sets and prove some of its properties. We use it to prove the compactness of some sequential of approximate solution of (1) under our hypothesis on A (t, u).

The paper in organized as follows.

- In section 2, we give some results on fuzzy normed spaces; we define a Hausdorff measure of non-compactness and prove some of its properties.
- In section 3, we recall some results on generalized Hukuhara differentiability and stat and prove our main results
- In section 4, we give some applications of our main results for the fuzzy partial differential equations:

$$\partial_t^{gH} u = f(t, x, u, \partial_{xx}^{gH} u(t, x)) \qquad (2)$$

And : $\partial_t^{gH} u = F(t, x, u, \partial_x^{gH} u(t, x))$ (3)

for $u \in X = (E_1, ||\cdot||)$ where $E_1 = \mathbb{R}_F$.

2 PRELIMINARIES

Let X be a non-empty set. *A* fuzzy subset of X is a mapping $u: X \longrightarrow [0, 1]$. where $u(x) = 0$ correspond to no membership, $0 < u(x) < 1$ to partial membership and $u(x) = 1$ to full membership; The α-level set $[u]_\alpha$ is defined as

$$[u]_\alpha = \{x \in X: u(x) \geq \alpha\} \text{ for each } \alpha \in [0, 1]$$

Let us denote by E^n the space of all fuzzy subsets of R^n satisfying the following conditions

(1) u maps R^n onto $I = [0, 1]$;

(2) $[u]_0$ is a bounded set of R^n;

(3) u is normal, that is: there exists at least one point $x_0 \in R^n$ such that $u(x_0) = 1$;

(4) $[u]_\alpha$ is a compact subset of R^n for all $\alpha \in I$; **(5)** u is fuzzy convex, that is

$$u(\lambda x + (1 - \lambda)y) \geq \min(u(x), u(y)) \qquad \text{for } \lambda \in I$$

(5) implies that $[u]_\alpha$ is convex subset of R^n (Lakshmikanthan [6]). We have the following representation Theorem.

Theorem 2.1. [9]

If $u \in E$, then

(i) $[u]_\alpha$ *is a non empty, compact and convex subset of* R^n *for all* $\alpha \in I$;

(ii) $[u]_{\alpha 2} \subseteq [u]_{\alpha 1}$ *for all* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$;

(iii) *if* (α_n) *is an increasing sequence converging to* α *then* $[u]_\alpha = \top_{n\geq 1} [u]_{\alpha n}$.

Let us consider the mappings $u^+_{\alpha i}, u^-_{\alpha i}:]0, 1] \longrightarrow R$ such that $-\infty < u_\alpha{}^-{}_i \leq u^+_{\alpha i} < +\infty$, and let $I_i^{\alpha_i} = \{x_i \in \mathbb{R}: u^-_{\alpha_i} \leq x_i \leq u^+_{\alpha_i}\}$. Then $I_i^{\alpha i} \subset R$ is a real interval and in view of Theorem 2.1 we can consider

$$[u]_\alpha = \prod_{i=1^n} I_i^{\alpha_i} \text{for all } 0 < \alpha_i \leq 1 \text{ and } [u]_0 = cl\left(\bigcap_{0<\alpha\leq 1} [u]_\alpha\right)$$

The supremum metric on E^n is defined by

$$D_\infty(u, v) = \sup_{0\leq\alpha\leq 1} \left(\max\left\{\|u^-_\alpha - v^-_\alpha\|_\infty, \|u^+_\alpha - v^+_\alpha\|_\infty\right)\right\}$$

where $\| w \|_\infty = \max_{1\leq i\leq n} |w_i| \; w \in R^n$.

D_∞ is a metric on E^n (see Taghavi et al.[8]). Let us consider the pseudo-norm

$$\|u\| = D_\infty(u, \tilde{0}),$$

Where

$$\vartheta(s) = \begin{cases} 0 if s = 0 \\ \neq 0 otherwise \end{cases}$$

Then $\| \cdot \|_n$ satisfies the following properties.

Theorem 2.2.

$\| \cdot \|_n: E^n \longrightarrow R+$ is such that *:*

(1) $\| \cdot \|_n = 0$ iff $u = \vartheta$.

(2) $\|\lambda \odot u\|_n = |\lambda| \cdot \|u\|_n$ for all $\lambda \in R$ and $u \in E^n$.

(3) $\| u \oplus v \|_n \leq \| u \|_n + \|v\|_n$, for all $u, v \in E^n$.

(4) $| \|u\|_n - \|v\|_n | \leq D_\infty(u, v)$, for all u, v ∈ E^n.

(5) $D_\infty(a \odot u,\ b \odot u) = |b-a| \cdot \|u\|_n$, for all a, b ∈ R and u ∈ E^n.

(6) $D_\infty(u,\ v) = \|u \ominus_{gH} v\|_n$ for all u, v ∈ E^n and $u \ominus_{gH} v$ is the generalized Hukuhara difference of u and v.

Proof.

(1) $\|u\|_n = D_\infty(u, \tilde{0}) = \sup_{0<\alpha\leq 1}(\max\{\|u^-_\alpha\|_\infty, \|u^+_\alpha\|_\infty\})$ and $D_\infty(u,\tilde{0}) = 0$ iff $\|u^-_\alpha\|_\infty = 0 = \|u^+_\alpha\|_\infty$, therefore $u = \tilde{0}$.

(2) For all *λ* ∈ R and *u* ∈ E^n,

$$\begin{aligned}\|\lambda \odot u\|_n &= D_\infty(u, \tilde{0}) \\ &= \sup_{0<\alpha\leq 1}\left(\max\left\{\|\lambda u^-_\alpha\|_\infty,\ \|\lambda u^+_\alpha\|_\infty\right)\right\} \\ &= |\lambda| \sup_{0\leq\alpha\leq 1}\left(\max\left\{\|u^-_\alpha\|_\infty,\ \|u^+_\alpha\|_\infty\right)\right\} \\ &= |\lambda| \cdot \|u\|_n.\end{aligned}$$

(3) For all *u, v* ∈ E^n,

$$\begin{aligned}\|u \oplus v\|_n &= \sup_{0<\alpha\leq 1}\left\{\max\left\{\|(u\oplus v)^-_\alpha\|_\infty,\ \|(u\oplus v)^+_\alpha\|_\infty\right)\right\} \\ &= \sup_{0<\alpha\leq 1}\left\{\max\left(\|u^-_\alpha + v^-_\alpha\|_\infty,\ \|u^+_\alpha + v^+_\alpha\|_\infty\right)\right\} \\ &\leq \sup_{0<\alpha\leq 1}\left\{\max\left(\|u^-_\alpha\|_\infty + \|v^-_\alpha\|_\infty,\ \|u^+_\alpha\|_\infty + \|v^+_\alpha\|_\infty\right)\right\} \\ &\leq \sup_{0<\alpha\leq 1}\left\{\max\left(\|u^-_\alpha\|_\infty,\ \|u^+_\alpha\|_\infty\right) + \max\left(\|v^-_\alpha\|_\infty,\ \|v^+_\alpha\|_\infty\right)\right\} \\ &= \|u\|_n + \|v\|_n.\end{aligned}$$

(4) For all *u, v* ∈ E^n,

$$| \|u\|_n - \|v\|_n | = |D_\infty(u, \tilde{0}) - D_\infty(v, \tilde{0})| \leq D_\infty(u, v)$$

(5) For all *a, b* ∈ R and *u* ∈ E^n,

$$\begin{aligned}D_\infty(a \odot u,\ b \odot u) &= \sup_{0<\alpha\leq 1}\left\{\max\left\{\|au^-_\alpha - bu^-_\alpha\|_\infty,\ \|au^+_\alpha - bu^+_\alpha\|_\infty\right)\right\} \\ &= \sup_{0<\alpha\leq 1}\left\{\max\left(|b-a|\|u^-_\alpha\|_\infty,\ |b-a|\|u^+_\alpha\|_\infty\right)\right\} \\ &= |b-a|\|u\|_n.\end{aligned}$$

(6) We show that $D_\infty(u,\ v) = \|u \ominus_{gH} v\|_n$. For all $u,\ v \in E^n$, $u \ominus_{gH} v \in E^n$ is equivalent to the existence of $c \neq \tilde{0}$ such that $u = v \oplus c$.

Hence $u^-_\alpha - v^-_\alpha \ncong \tilde{0}$ and $u^+_\alpha - v^+_\alpha \ncong \tilde{0}$.

Hence

$$\begin{aligned}\|u \ominus_{gH} v\|_n &= \sup_{0<\alpha\leq 1}\left\{\max\left(\|u^-_\alpha - v^-_\alpha\|_\infty,\ \|u^+_\alpha - v^+_\alpha\|_\infty\right)\right\} \\ &= D_\infty(u,\ v).\end{aligned}$$

Let us consider E^n with the pseudo-norm $\|\cdot\|_n$ then the fuzzy space (E^n, $\|\cdot\|_n$) is not linear.

On the other hand (E^n, D_∞) is a complete metric space. Let us denote by We define an $X = (E^n,\ \|\cdot\|_n)$. open ball in *X* with center *a* ∈ *X* and radius *R* by

$$B(a; R) = \{u \in E^n \mid D_\infty(u, a) < R\}$$

The closed ball is defined by

$$B^{-}(a; R) = cl(B(a; R))$$

Definition 2.3.

A subset $B \subset X$ is uniformly bounded if there exists a constant β such that

$$B \subset B(\tilde{0}; \beta)$$

Let $\mathcal{B} \subset X$ be the set of all uniformly bounded subsets of X, we define the diameter of $B \subset X$ by

$$diam B = \sup \{\|u \ominus_{gH} v\|_n \mid \quad u \in B \; and \; v \in B\}$$

Definition 2.4.

Let *$\{B(a_i; \varepsilon_i)\}_{i\in I}$, $a_i \in E^n$, $\varepsilon \in R_+$, $\varepsilon_i > 0\}$* be a family of opens balls in *X* and let $B \subset X$ be such that $B \subset \bigvee_{i\in I} B(a_i; \varepsilon_i)$, then $\{B(a_i; \varepsilon_i)\}_{i\in I}$ is a ε–cover of B.

In general, if $\{U_i\}_{i\in I} \subset X$ is a family in *X* such that $B \subset \bigvee_{i\in I} U_i$, then $\{(U_i)\}_{i\in I}$ is a cover of *B.*

Since *X* is not a compact space, we define a fuzzy non compactness measure:

Definition 2.5.

Let $B \in \mathcal{B}$ be a uniformly bounded set of *X*. The mapping $\beta : \mathcal{B} \longrightarrow \mathbb{R}_+$ defined by *β(B) = inf{d > 0 such that B is covered by a finite number of fuzzy subset of diameter less than d}* is a Hausdorff measure of non compactness.

Proposition 2.6.

Let $\beta : \mathcal{B} \longrightarrow R_+$ be the Hausdorff measure of non compactness of the Definition 2.5. Then

- **a)** $\beta(B) = 0$ *iff B* is (relatively) compact.
- **b)** β is a semi-norm, that is
  - **(i)** $\beta(\lambda \odot B) = |\lambda|\beta(B)$ for all $B \in \mathcal{B}$ and $\lambda \in R$.
  - **(ii)** $\beta(B_1 \oplus B_2) \leq \beta(B_1) + \beta(B_2)$ *for all* $B_1 \in \mathcal{B}$ *and* $B_2 \in \mathcal{B}$.
- **c)** *If* $B_1 \subset B_2$ *then* $\beta(B_1) \leq \beta(B_2)$.
- **d)** $\beta(B_1 \vee B_2) = \max(\beta(B_1), \beta(B_2))$
- **e)** *β is continuous with respect to D_∞.*

Proof.

We observe that (c), (d) and (e) are easy consequences of the definition of *β* and properties of fuzzy sets. So we prove (a) and (b).

(a) $\beta(B) = 0$ if and only if $\|u \ominus_{gH} v\|_n = 0$ for all *u* and $v \in B$ that is $D_\infty(u, v) = 0$ which means that any open ball centered at *u* contains *v,* for all $v \in B$. There for, one can find a finite family of sets $B_1 = B(u; r)$, $B_2 = \emptyset, \ldots, B_p \subset \emptyset$ such that $B = \bigvee_{i=1}^{p} B_i$. Hence (a).

(b) (i) By the property (2) of $||\cdot||_n$, we have

$$\|\lambda \odot u\|_n = |\lambda| \|u\|_n$$

Let $B_1, B_2, \cdots, B_p \subset X$ and $\varepsilon > 0$ be such that $B = \bigvee_{i=1}^{p} B_i$ and $diam B_i \leq \beta(B) + \varepsilon$.

Since $\|\lambda \odot u \ominus_{gH} \lambda \odot v\|_n = |\lambda| \|u \ominus_{gH} v\|_n$, we have

$$|\lambda| \sup_{B_1} \{\|u \ominus_{gH} v\|_n\} = \sup_{B_1} \{\|\lambda \odot u \ominus_{gH} \lambda \odot v\|_n\}$$
$$= diam(\lambda \odot B_i).$$

Therefore

$$diam(\lambda \odot B_i) = |\lambda| diam B_i \leq |\lambda|(\beta(B) + \varepsilon)$$

Hence $\beta(\lambda B) \leq |\lambda|\beta(B)$ since ε is arbitrary.

If λ 6= 0, then

$$\beta(B) = \beta\left((\frac{1}{\lambda}.\lambda)B\right)$$
$$= \beta\left(\frac{1}{\lambda}.(\lambda \odot B)\right)$$
$$\leq \frac{1}{|\lambda|}\beta(\lambda \odot B).$$

Therefore

$$|\lambda|\beta(B) \leq \beta(\lambda \odot B)$$

Hence (b) is proved.

(ii) Let $\{S_i\}_{i=1}^n$ be a finite family of subsets of *X* such that $B_1 \subset \bigvee_{i=1}^n S_i$ and $diam S_i \leq \beta(B_1) + \frac{\varepsilon}{2}$, and let $\{T_j\}_{j=1}^n$ another family of subsets of *X* such that $B_2 \subset \bigvee_{j=1}^n T_j$ and $diam T_j \leq \beta(B_2) + \frac{\varepsilon}{2}$. Since $B_1 \oplus B_2 \subset \bigvee_{i,j=1}^n S_i \oplus T_j$, by the property (3) of $||\cdot||_n$ we have

$$diam\,(S_i \oplus T_j) \leq diam S_i + diam T_j$$
$$\leq \beta(B_1) + \frac{\varepsilon}{2} + \beta(B_2) + \frac{\varepsilon}{2}$$
$$= \beta(B_1) + \beta(B_2) + \varepsilon.$$

Hence (ii) is proved, since ε is arbitrary.

3 THE CAUCHY PROBLEM

3.1 GENERALIZED HUKUHARA DERIVATIVE

Definition 3.1. [2]

Let u, v ∈ X

$$u \ominus_{gH} v = w \textit{ iff} \begin{cases} (i) & u = v \oplus w \\ (ii) & v = u \oplus (-1) \odot w. \end{cases}$$

In the term of α−level set, we have

(a) *For* $n = 1,\ [u \ominus_{gH} v]_\alpha = [\min\{u_\alpha^- - v_\alpha^-,\ u_\alpha^+ - v_\alpha^+\},\ \max\{u_\alpha^- - v_\alpha^-,\ u_\alpha^+ - v_\alpha^+\}]$.

(b) In E^n, the existence of $[u]_\alpha \ominus_{int} [v]_\alpha$ does not imply $u \ominus_{gH} v$ in general.

Definition 3.2. [6]

A mapping $F : I \longrightarrow E^n$ is Hukuhara differentiable at $t_0 \in I$ if there exists $F^0(t_0) \in E^n$ such that

$$\lim_{h\to 0}\frac{F(t_0+h)\ominus_H F(t_0)}{h} \quad and \quad \lim_{h\to 0}\frac{F(t_{0)}\ominus_H F(t_0+h)}{h}$$

exist and are equal to $F'(t_0)$. Limits are taken in (E^n, D_∞).

From Definition 3.2, one gets the differential of the multivariate mapping F_α, $\alpha \in [0, 1]$ given by

$$DF_\alpha(t) = [F^0(t)]_\alpha$$

On the other hand, we have

Theorem 3.3. [6]

Let $F : I \longrightarrow E^n$

(a) For all $t \in I$, there exists $\beta > 0$ such that the H-difference $F(t+h)\ominus_H F(t)$ and $F(t)\ominus_H F(t-h)$ exist for all $0 \le h < \beta$.

(b) The set-valued mapping F_α, $\alpha \in [0, 1]$ are uniformly differentiable, that is, there exists DF_α such that, there exists $\delta > 0$, and

$$D_\infty(F_\alpha(t+h)\ominus_{int} F_\alpha(t),\ D_\infty F_\alpha(t)) < \varepsilon$$

and

$$D_\infty(F_\alpha(t)\ominus_{int} F_\alpha(t-h),\ D_\infty F_\alpha(t)) < \varepsilon$$

for all $\varepsilon > 0$ and $0 \le h < \delta$.

That is

$$\lim_{h\to 0}\|\frac{F_\alpha(t+h)\ominus_{int} F_\alpha(t)\ominus_H hDF_\alpha(t)}{h}\|_n = 0 \quad (4)$$

and

$$\lim_{h\to 0}\|\frac{F_\alpha(t)\ominus_{int} F_\alpha(t-h)\ominus_H hDF_\alpha(t)}{h}\|_n = 0. \quad (5)$$

Theorem 3.4. [1]

Let $F: I \longrightarrow E^1$ and denote $F_\alpha(t) = [f_\alpha^-(t), f_\alpha^+(t)]$, $\alpha \in [0, 1]$. Then F is differentiable if f_α^-, f_α^+ are differentiable and, we say that:

(a) F is ((i) – gH)–differentiable at $t_0 \in I$, if

$$[F'(t_0)]_\alpha = \left[(f_\alpha^-)'(t_0),\ (f_\alpha^+)'(t_0)\right];\ \alpha \in [0, 1] \quad (6)$$

(b) F is ((ii) – gH)–differentiable at $t_0 \in I$, if

$$[F'(t_0)]_\alpha = \left[(f_\alpha^+)'(t_0),\ (f_\alpha^-)'(t_0)\right];\ \alpha \in [0,\ 1] \quad (7)$$

Definition 3.5. [1]

A point $t_0 \in I$ is a switching point for F^0 if in any neighborhood V of t_0 in the interior of I, there exist two points t_1 and t_2 such that $t_1 < t_0 < t_2$ and I_1 (6) holds at t_1 and (7) doesnot hold at t_2 or (7) holds and (6) doesnot hold.

II_1 (7) holds at t_1 and (6) doesnot hold at t_2 or (6) holds and (7) doesnot hold.

Theorem 3.6. [6]

Assume F : I⟶E^1 differentiable and doesnot have a switching point then, if F is integrable over I, we have

$$\int_a F'(s)ds = F(b) \ominus_H F(a) \ for\ I = [0,\ 1] \subset \mathbb{R}$$

Proof. For all α ∈ [0, 1] fixed, if there is no switching point, then we may prove that

$$F_\alpha(b) = F_\alpha(a) \oplus_{int} \int_a^b DF_\alpha(s)\ or\ F_\alpha(a) = F_\alpha(b) \ominus_{int} \int_a^b DF_\alpha(s) \quad (8)$$

3.2 Main results

Let $X = (E^1, ||\cdot||_1)$ and consider the fuzzy differential problem

$$u'_{gH} = A(t, u)\ t \in I \quad (9)$$
$$u(t_0) = u_0,$$

Where

$A : I \times X \longrightarrow X$ such, $I \subset R$ and $t_0 \in I$ an open subset of R.

$A(t, \cdot) : X \longrightarrow X$ is continuous.

$A(\cdot, u) : I \longrightarrow X$ is strongly measurable.

We consider the Cauchy problem (9) under the following hypothesis on A :

(H1) $||A(t, u)||_1 \leq C||u||_1 + M$, where $C > 0$, and $M > 0$ are real.

(H2) There exists $K > 0$ such that $\beta(A(I \times B)) \leq K\beta(B)$ for any $B \in B$.

(H3) There is no switching point.

Our main result can be formulate as follows.

Theorem 3.7.

Assume (H1)-(H3) hold, then (9) has at least one solution.

We shall need some preliminary lemmas.

Lemma 3.8.

Assume (H1) and (H3) hold, then there exists an approximate solution of (9) in $[t_0, t_0 + \alpha]$ for some α > 0 small.

Proof.

Without lost of generality, let $t_0 = 0$, $u(0) = \vartheta$; and $B \subset B(\vartheta, \delta)$ for some $\delta > 0$, be a bounded set in and X, $a_0 = \min\{\alpha, \frac{\delta}{M+C+1}\}$ $T_0 = [0, a_0] \subset [0, \alpha]$. Let us partition T_0 into subintervals $0 < t_1 < t_2 < \cdots < t_N = a_0$. For any $t \in [t_j, t_{j+1}]$, $j = 1, 2, ..., N$, $(t - t_j) \odot A(t, u(t))$ is well defined, measurable on t for every $u \in B$ and continuous on $u \in B$ for a.e. $t \in]t_j, t_{j+1}[= T_0^j \subset T_0$. Therefore, we can define the sequence $(u_n(t))_{t\in I}$, by:

$$u_n(t) = \begin{cases} \vartheta & \text{if } t \leq 0 \\ u_n(t_k) \oplus (t - t_k) \odot A(t_k, u_n(t_k)) & \text{if } s = 0,\ t \in [t_k, t_{k+1}] \\ \vartheta & \text{if } t > a_0 \end{cases}$$

and for $j = 1, 2, ..., N$ we have $u_n(t_k) \in B(\vartheta, \delta)$. Clearly $T_0 = [0, a_0] \neq \emptyset$ since $0 \in T_0$. We observe that

$$\int_{t_k}^{t} A(t_k,\, u_n(t_k))ds = (t - t_k) \odot A(t_k,\, u_n(t_k)) \qquad (10)$$

We may define t_k by $t_k = \frac{ka_0}{N},\ k = 1,\, 2,\, \ldots, N$ to partition T_0.

We first have to prove that:

(i) u_n is uniformly continuous on $]-\infty, a_0]$.

(ii) u_n is *piecewise* derivable on T_0.

(i) For all $t, t^0 \in T_0$, we have

$$\|u_n(t) \ominus_{gH} u_n(t')\|_1 = \sup_{0<\alpha\leq 1} \{\max\{\|(u_n)^-_\alpha(t) - (u_n)^-_\alpha(t')\|_\infty, \|(u_n)^+_\alpha(t) - (u_n)^+_\alpha(t')\|_\infty\}\}$$

Let us note by

$$D^+_\infty(u,\, v) = \|u^+_\alpha - v^+_\alpha\|_\infty \text{ and } D^-_\infty(u,\, v) = \|u^-_\alpha - v^-_\alpha\|_\infty$$

Hence

$$\begin{aligned} D_\infty^+(u_n(t), u_n(t^0)) &= ||(u_n)^+_\alpha(t) - (u_n)_\alpha^+(t_k) + (u_n)^+_\alpha(t_k) - (u_n)^+_\alpha(t^0)\,||_\infty \\ &= ||\,(t - t_k)\, A^+_\alpha(t_k, u_n(t_k)) + (t_k - t^0)\, A^+_\alpha(t_k, u_n(t_k))||_\infty \\ &= ||\,(t - t^0)\, A^+_\alpha(t_k, u_n(t_k))\,||_\infty \\ &= |t - t^0|\,||A^+_\alpha(t_k, u_n(t_k))\,||_\infty. \end{aligned}$$

Similarly

$$D^-_\infty(u_n(t),\, u_n(t')) = |t - t'|\|A^-_\alpha(t_k,\, u_n(t_k))\|_\infty$$

Therefore

$$\begin{aligned} \|u_n(t) \ominus_{gH} u_n(t')\|_1 &= \sup_{0<\alpha<1} \{\max\{|t - t'|\|A^-_\alpha(t_k, u_n(t_k))\|_\infty, |t - t'|\|A^+_\alpha(t_k, u_n(t_k))\|_\infty\}\} \\ &= |t - t'|\|A(t_k,\, u_n(t_k))\|_1 \\ &\leq |t - t'|(C\rho + M). \end{aligned}$$

For $\delta < \frac{\varepsilon}{C\rho + M},\ |t - t'| < \delta$, we have $\|u_n(t) \ominus_{gH} u_n(t')\|_1 < \varepsilon$.

Thus $u_n(t)$ is uniformly continuous on T_0. Since $u_n(t) = \tilde{0}$ for all $t \leq 0$, we have, $u_n(t)$ uniformly continuous on $]-\infty, a_0]$.

(ii) For all $t \in\,]t_k,\ t_{k+1}[\ = \overset{\circ}{T}{}_0^k$, consider $h > 0$ small enough such that $t + h$ and $t - h \in \overset{\circ}{T}{}_0^k$, then $u_n(t + h) \ominus_{gH} u_n(t)$ is well defined on $\overset{\circ}{T}{}_0^k$ and

$$\begin{aligned} &D_\infty(u_n(t + h) \ominus_{gH} u_n(t),\, h \odot A(t_k,\, u_n(t_k))) \\ &= \sup_{0<\alpha<1} \{\max\{\|(u_n)^-_\alpha(t + h) - (u_n)^-_\alpha(t) - hA^-_\alpha(t_k,\, u_n(t_k))\|_\infty \\ &\|(u_n)^+_\alpha(t + h) - (u_n)^+_\alpha(t) - hA^+_\alpha(t_k,\, u_n(t_k))\|_\infty\}\}. \end{aligned}$$

We have

$$\|(u_n)^\pm_\alpha(t + h) - (u_n)^\pm_\alpha(t) - hA^\pm_\alpha(t_k, u_n(t_k))\|_\infty = \|[(t + h - t) - h]A^\pm_\alpha(t_k, u_n(t_k))\|_\infty = 0$$

Hence

$$\lim_{h\to 0}\|\frac{(u_n)^{\pm}_{\alpha}(t+h)-(u_n)^{\pm}_{\alpha}(t)-hA^{\pm}_{\alpha}(t_k,\,u_n(t_k))}{h}\|_\infty=0$$

So that

$$((u_n)^{\pm}_{\alpha})'(t)=A^{\pm}_{\alpha}(t_k,\,u_n(t_k))$$

Similarly

$$((u_n)^{\mp}_{\alpha})'(t)=A^{\mp}_{\alpha}(t_k,\,u_n(t_k))$$

Therefore, assuming that there is no switching point at t_k and t_{k+1} for all $1 \le k \le N$, then we have

$$(u_n)'_{gH}(t)=A(t_k,\,u_n(t_k)) \tag{11}$$

Lemma 3.9.

Assume (H1) and (H2) hold. If $u_n(t)$ is an approximate solution of (9), piecewise differentiable such that, there is no switching point at t_k and t_{k+1} for $1 \le k \le N-1$, then

$$\|u_n(t)\ominus_H u_n(t_0)\ominus_H \int_{t_0}^{t} A(s,\,u_n(s))ds\|_1\le\varepsilon|t-t_0|$$

Proof.

Let $t_0 = 0$, $t > 0$ and a partition of $T = [0, t]$ defined by $0 < t_1 < t_2 \cdots < t_N = t$ such that $t_k=\frac{kt}{N}$, for $k=1,\,2,\cdots,N$, for $N > 0$ large enough, and $u_n(t)$ derivable on $]t_j, t_{j+1}[$, $j = 1, 2\cdots, N-1$. Assume that $(u_n'(t))$ is bounded and measurable, which is possible by (H1), the definition of $u_n(t)$ and (11) and $u_n'(t)$ has no switching point. Then by the Aumann definition of fuzzy integral we have

$$un(tk+1) = un(tk) \oplus \int_{t_k}^{t_{k+1}}(un)gH(s)ds$$

And by Lemma 3.8, and continuity of integral we have

$$\begin{aligned}
&\|u_n(t_{k+1})\ominus_H u_n(t_k)\ominus_H \int_{t_k}^{t_{k+1}} A(s,\,u_n(s))ds\|_1\\
&=\|\int_{t_k}^{t_{k+1}}((u_n)'_{gH}(s))\ominus_H A(s,\,u_n(s)))ds\|_1\\
&\le\int_{t_k}^{t_{k+1}}\|((u_n)'_{gH}(s))\ominus_H A(s,\,u_n(s)))\|_1 ds\le\varepsilon\int_{t_k}^{t_{k+1}}ds\\
&=\varepsilon|t_{k+1}-t_k|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|u_n(t)\ominus_H u_n(0)\ominus_H \int_0^t A(s,\,u_n(s))ds\|_1 &=\|\int_0^t((u_n)'_{gH}(s))\ominus_H A(s,\,u_n(s)))ds\|_1\\
&=\|\sum_{k=1}^{N-1}\int_{t_k}^{t_{k+1}}((u_n)'_{gH}(s))\ominus_H A(s,\,u_n(s)))ds\|_1\\
&\le\sum_{k=1}^{N-1}\int_{t_k}^{t_{k+1}}\|((u_n)'_{gH}(s))\ominus_H A(s,\,u_n(s)))\|_1 ds\\
&\le\varepsilon\sum_{k=1}^{N-1}\int_{t_k}^{t_{k+1}}ds=\varepsilon\sum_{k=1}^{N-1}|t_{k+1}-t_k|\le\varepsilon|t|,
\end{aligned}$$

for all t such that $]0, t[\subset [0, a_0]$.

Proof of Theorem 3.7

The standard way to prove existence of solution of (9) by mean of approximate solution is to prove that

1. $(u_n(t))_{n\geq 1}$ is uniformly bounded and equi-continuous on $[0, a_0]$.
2. Using Ascoli-Arzela Theorem, $(u_n(t))_{n\geq 1}$ converges or has a subsequence converging to a solution of (9).

Unfortunately, X is not a compact space and (2^0) is not sufficient to have convergence. We must use a non-compactness argument.

Consider the differential equation

$$(u_n)'_{gH}(t) = A(t,\, u_n(t))$$

By (H3) and Theorem 3.6, we have

$$u_n(t) \ominus_H u_n(0) = \int_0^t (u_n)'_{gH}(s)ds = \int_0^t A(s,\, u_n(s))ds$$

and

$$\|u_n(t) \ominus_H u_n(0)\|_1 \leq \int_0^t \|A(s,\, u_n(s))\|_1 ds.$$

By (H1), there exist $C > 0$ and $M > 0$ reals such that

$$\int_0^t \|A(s,\, u_n(s))\|_1 ds \leq C\int_0^t \|u_n(s)\|_1 ds + a_0 M.,\ \forall \varepsilon > 0$$

Since $u_n(0) = \vartheta$, we have

$$\|u_n(t)\|_1 \leq a_0(M+\varepsilon) + C\int_0^t \|u_n(s))\|_1 ds,\ \forall \varepsilon > 0$$

By the Gronwall inequality, we have

$$||u_n(t)||_1 \leq a_0(M+\varepsilon)e^{Ct} \leq a_0(M+\varepsilon)e^{aC} = M_0$$

Therefore $(u_n(t))_{n\geq 1}$ is uniformly bounded. For $t, t^0 \in [0, a_0]$, we have

$$\begin{aligned}\|u_n(t) \ominus_H u_n(t')\|_1 &= \|\int_{t'}^t A(s,\, u_n(s))ds\|_1\\ &\leq \int_{t'}^t \|A(s,\, u_n(s))\|_1 ds\\ &\leq |t-t'|(CM_0+M).\end{aligned}$$

For $\delta > 0$ such that $|t - t^0| < \delta$ and $\delta < \frac{\varepsilon}{CM_0+M}$, we have

$$\|u_n(t) \ominus_H u_n(t')\|_1 < \varepsilon.$$

Thus $(u_n(t))_{n\geq 1}$ is equi-continuous.

Let $N > 0$ in N, $t \in [0, a_0]$ and define $B_N(t) = \{u_n(t) : n \geq N\}$. Then $B_N(0) = \vartheta$ and $B_N([0, a_0])$ is bounded. Let $w(t) = \beta(B_N(t))$, then $w(t)$ is continuous.

Indeed, by the property (b)(ii) of β, we have, for all $n \geq N$

$$\begin{aligned}\beta(\{u_n(t)\}) - \beta(\{u_n(s)\}) &\leq \beta(\{u_n(t) \ominus_H u_n(s)\}) \\ &= \beta(|t-s| \odot A(s,\, u_n(s)) \\ &= \beta(|t-s| \odot A(s,\, B_N(s)) \\ &\leq |t-s| K\beta(B_N([0,\, a_0])).\end{aligned}$$

Since $B_N([0, a_0])$ is a bounded set, there exists $d_0 > 0$ such that $\beta(B_N([0, a_0])) \leq d_0$. Therefore

If $|w(t) - w(s)| \leq |t - s| Kd_0$

Hence $\delta < \dfrac{\varepsilon}{Kd_0 + 1}$, and $|t - s| < \delta$, we have $|w(t) - w(s)| < \varepsilon$, *for all* $\varepsilon > 0$

Now $\dfrac{w(t+h) - w(t)}{h}$ and $\dfrac{w(t) - w(t-h)}{h}$ are well defined for all $h > 0$ such that $[t - h, t + h] \subset [0, a_0]$.

Let

$$D_{-}w(t) = \lim_{h\to 0^+} \inf \frac{w(t) - w(t-h)}{h}.$$

Then

$$\begin{aligned}D_{-}w(t) &= \lim_{h\to 0^+} \inf \beta(\frac{1}{h} \odot [u_n(t) \ominus_H u_n(t-h)]), \quad \forall\, n \geq N. \\ &= \lim_{h\to 0^+} \inf_{n\geq N} \{\frac{1}{h}\beta(h \odot A(t-h,\, u_n(t-h)))\} \\ &\leq \beta(\{(u_n)'_{gH}(t)\}), \quad \forall\, n \geq N.\end{aligned}$$

Let us choose $h > 0$ such that $I_h = [t - h, t]\subset [0, a_0]$ and $\beta(u_n(I_h)) < \varepsilon_n$ for all $n \geq N$ and $\varepsilon_n \to 0$ as $n \to \infty$, then

$$\beta(B_N(I_h)) \leq \beta(A(I_h \times B_N(I_h) \oplus u_n(I_h)) \leq \beta(A(I_h \times B_N(I_h))) + \beta(u_n(I_h)) \leq K\beta(B_N(I_h) + \varepsilon_n$$

Using the equicontinuity of $(u_n(t))_{n\geq 1}$, we have $B_N(I_h) \to B_N(t)$ as $h \to 0^+$ with respect to the Hausdorff distance D_∞. Therefore :

$$D_{-}w(t) \leq Kw(t) + \varepsilon_n, \text{ for all } n \geq N$$

Integrating over $[0, t]$ for all $t \in [0, a_0]$, and taking into account that $w(0) = 0$, we have

$$w(t) \leq \int_0^t Kw(s)ds + \varepsilon_n a_0$$

Applying the Gronwall inequality, we have

$$w(t) \leq \varepsilon_n a_0 e^{Ka0}$$

Since $\varepsilon_n \to 0$ as $n \to \infty$, we have

$$\beta(\{u_n(t)\}) \to 0 \text{ as } n \to \infty$$

Thus $(u_n(t))_{n\geq 1}$ is relatively compact. Taking a subsequence, if necessary, we may assume that $(u_n(t))_{n\geq 1}$ converges to $u(t) \in E^1$.

Let $B \subset B(\hat{0}, \rho)$ be a bounded set, using (H1) in B, we have

$$||A(s, u_n(s))||_1 \leq C||u_n(s)||_1 + M$$

$$\leq c\rho + M = M_\rho$$

for all $n \geq N$, and $s \in [0, a_0]$ such that $u_n(s) \in B$. By (9) we have

$$u_n(t) = u_0 \oplus \int_0^t A(s,\, u_n(s))ds.$$

Using the fuzzy dominate convergence (see e.g. [10]) we have

$$u(t) = u_0 \oplus \int_0^t A(s,\, u(s))ds, \quad t \in [0,\, a_0]$$

Assume that

(H4) $A : I \times X \dashrightarrow X$ is locally-Lipschitz continuous on u. We have

Theorem 3.10.

Assume that A satisfies (H1), (H3) and (H4), then (9) has at least one solution on $[0, a_0]$.

Proof.

By Theorem 3.7, it suffices to show that (H4) implies (H2). Let $L > 0$ be the Lipschitz constant of A and B be a bounded set in X. Let $\{E_j\}_{j=1}^N$ a finite covering of B of diameter d_j, and $d = \inf_j d_j$. Then, for all $u_1,\ u_2\ \in B,\ \|u_1(t) \ominus_H u_2(t)\|_1 \leq d$ and $\|A(t,\, u_1(t)) \ominus_H A(t,\, u_2(t)\|_1 \leq L\|u_1(t) \ominus_H u_2(t)\|_1 \leq Ld.$

Hence $\beta(A(I \times B)) \leq L\beta(B)$.

Therefore, if we choose $L = K$, we have $\beta(A(I \times B)) \leq K\beta(B)$. Hence (H2) and we are done.

4 APPLICATIONS TO FUZZY NONLINEAR EVOLUTION EQUATIONS

4.1 GENERALIZED HUKUHARA PARTIAL DERIVATIVES

Definition 4.1. [1]

A fuzzy valued function of two variables is a relation that assigns to each ordered pair of real number in a set $D \subset R^2$, a unique fuzzy number denoted by f (x, t). The set D is then the domain of f and $R_f \subset E^1$ such that $R_f = \{f(x, t) \mid (x, t) \in D\}$ is the range of f.

The α–level set for f is represented by $[f(x,t)]_\alpha = [f_\alpha^-(x,t),\ f_\alpha^+(x,t)]$, for all $\alpha \in [0, 1]$ and $(x, t) \in D$. Note that f_α^- and $f_\alpha^+ \in \mathbb{R}$.

Definition 4.2. [1]

Let $f : D \longrightarrow E^1$ be a fuzzy valued function of two variables. We say that $L \in E^1$ is the limit of f(x,t) as $(x,t) \longrightarrow (x_0,t_0)$ if for every $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that if $(x, t) \in D$ with $||(x, t) - (x_0,t_0)|| < \delta$, then $\|f(x,\, t) \ominus_{gH} L\|_1 < \varepsilon.$

Definition 4.3. [1]

A fuzzy valued function $f : D \longrightarrow E^1$ is fuzzy-continuous at $(x_0,t_0) \in D$ if lim(x,t)→(x0,t0) f(x, t) = f(x0, t0). f is fuzzy-continuous on D if it is fuzzy-continuous at each point of D.

Definition 4.4. [1]

Let $f : D \longrightarrow E^1$ be a fuzzy valued function. If $f_\alpha^-(x, t)$ and $f_\alpha^+(x, t)$ are differentiable with respect to x and t, then we say that f(x, t) is

(I − PgH)−derivable with respect to x if

$$\partial_{x_{I-gH}} [f(x,\, t)]_\alpha = \left[\partial_x f_\alpha^-(x,\, t),\ \partial_x f_\alpha^+(x,\, t)\right]$$

(I – PgH)–derivable with respect to t if

$$\partial_{t_{I-gH}}\left[f(x,\,t)\right]_{\alpha}=\left[\partial_t f_{\alpha}^{-}(x,\,t),\ \partial_t f_{\alpha}^{+}(x,\,t)\right]$$

(II– PgH)–derivable with respect to x if

$$\partial_{x_{I-gH}}\left[f(x,\,t)\right]_{\alpha}=\left[\partial_x f_{\alpha}^{+}(x,\,t),\ \partial_x f_{\alpha}^{-}(x,\,t)\right]$$

(II – PgH)–derivable with respect to t if

$$\partial_{t_{I-gH}}\left[f(x,\,t)\right]_{\alpha}=\left[\partial_t f_{\alpha}^{+}(x,\,t),\ \partial_t f_{\alpha}^{-}(x,\,t)\right]$$

We assume that $\partial_x f(x, t)$ is $(P-gH)$–derivable at every $(x, t) \in D$ without switching point throughout the section. We have the following definition.

Definition 4.5. [1]

$\partial_{xgH} f(x, t)$ is

(I – PgH)–derivable with respect to x if for all (x, t) ∈ D

$$\partial_{xx_{I-gH}}\left[\partial_x f(x,\,t)\right]_{\alpha}=\left[\partial_{xx} f_{\alpha}^{-}(x,\,t),\ \partial_{xx} f_{\alpha}^{+}(x,\,t)\right] \tag{12}$$

(II – PgH)–derivable with respect to x if for all (x, t) ∈ D

$$\partial_{xx_{II-gH}}\left[\partial_x f(x,\,t)\right]_{\alpha}=\left[\partial_{xx} f_{\alpha}^{+}(x,\,t),\ \partial_{xx} f_{\alpha}^{-}(x,\,t)\right] \tag{13}$$

Lemma 4.6. [1]

Let f: D ⟶ E^1 be a fuzzy-continuous. If f is (P – gH) –derivable with respect to t, without switching point on [a, τ] ⊂ [a, b] ⊂ $\mathbb{R}$ with fuzzy-continuous derivative, then

$$\int_a^{\tau}\partial_{s_{P-gH}} f(x,\,s)ds=f(x,\,\tau)\ominus_{gH} f(x,\,a) \tag{14}$$

4.2 CAUCHY-KOWALESKYA THEOREMS

Let $u : D \subset R \times R+ \longrightarrow E^1$ be a fuzzy valued function, and define a fuzzy differential relation on *D* by

$$\partial_{t_{gH}}u(x,\,t)=k\odot\partial_{xx_{gH}}u(x,\,t)\oplus g\left(u(x,\,t)\right) \tag{15}$$

where $g: E^1 \longrightarrow E^1$ is a nonlinear fuzzy function with fuzzy variable. We use the following initial value

$$u(x, 0) = u_0 \in E^1$$

Let $X=(E^1,\,\|\cdot\|)$, where k*u*k = $D_\infty(u, \tilde{0})$. We define a fuzzy operator A by

$$D(A) = \{u \in X: \partial_{xxgH}u,\ g(u) \in X\}$$

with

$$Au(x,\,t)=k\odot\partial_{xx_{gH}}u(x,\,t)\oplus g(u(x,\,t)),\ \ k\in\mathbb{R}\setminus\{0\}$$

We consider the Cauchy Problem

$$\dot{u}(t) = Au(t)\ t \in I \subset R,\ I =]0, a_0[,\ a_0 > 0 \quad (16)$$

$$u(0) = u_0,$$

where $\dot{u}(t) = \partial_{tgH}u(x, t)$.

We assume that g satisfies the following assumptions.

(G1) $g : X \longrightarrow X$ is locally Lipschitz-continuous.

(G2) There exists $M_0 > 0$ and $C > 0$ such that

$$||g(u)|| \leq C||u|| + M_0.$$

Lemma 4.7.

If (G1) and (G2) hold, then (H1) and (H2) are satisfied.

Proof.

Set $k = 1$ in (15), then we have

$$Au(t) = A(t, u(x, t))$$
$$= D_{xx}u \oplus g(u),$$

where $D_{xx} = \partial_{xxgH}$. The α–level set of $Au(t)$ is given by $[Au(t)]_\alpha = [A_\alpha^- u(t),\ A_\alpha^+ u(t)]$ where

$$A_\alpha^- u(t) = (D_{xx}u \oplus g(u))_\alpha^- = (D_{xx}u)_\alpha^- + g_\alpha^-(u)$$

and

$$A_\alpha^+ u(t) = (D_{xx}u \oplus g(u))_\alpha^+ = (D_{xx}u)_\alpha^+ + g_\alpha^+(u)$$

We observe that $(D_{xx}u)_\alpha^-$ and $(D_{xx}u)_\alpha^+$ are linear and continuous for all $\alpha \in [0, 1]$ and $u \in B$, where $B \subset B(\delta, \rho)$ is a bounded set. Let C_0^- and C_0^+ be such that $\|(D_{xx}u)_\alpha^-\|_\infty \leq C_0^-$ and $\|(D_{xx}u)_\alpha^+\|_\infty \leq C_0^+$.

Therefore

$$\|D_{xx}u\| = \sup_\alpha \max\left(\|(D_{xx}u)_\alpha^-\|_\infty,\ \|(D_{xx}u)_\alpha^+\|_\infty\right) \leq C_0$$

Where $C_0 = \max\{C_0^-,\ C_0^+\}$, and by (G1), we have

$$||Au(t)|| \leq ||D_{xx}u|| + ||g(u)|| \leq C_0 + C||u|| + M_0$$

Let $M = C_0 + M_0$, then $M > 0$ and $||Au(t)|| \leq C||u|| + M$. Hence (H1) holds.

Let $C > 0$ be the Lipschitz constant of g and $B \subset X$ be a bounded set with finite cover $\{E_j\}_{j=1}^N$ of diameter d_j, $j = 1, 2..., N$. Let $d = \inf d_j$, then for all u_1 and $u_2 \in B$, we have

$$\|u_1(t) \ominus_{gH} u_2(t)\| \leq d$$

and

$$\|g(u_1) \ominus_{gH} g(u_2)\| \leq C\|u_1(t) \ominus_{gH} u_2(t)\| \leq Cd.$$

Since $||D_{xx}u|| \leq C_0$ for all $u \in B$, we have

$$\|D_{xx}u_1 \ominus_{gH} D_{xx}u_2\| = \|D_{xx}(u_1 \ominus_{gH} u_2)\| \leq C_0$$

where C_0 depends on d. Let us choose $C_0 > 0$ such that $C_0 = \rho d$. Then

$$\|Au_1(t) \ominus_{gH} Au_2(t)\| \leq \rho d + Cd = (\rho + C)d.$$

Hence $\beta(A(B))$ is well defined and $\beta(A(B)) \leq K\beta(B)$, where $K = \rho + C > 0$. Hence, (H2) holds.

Using Theorem 3.7 and Lemma 4.7, we have:

Theorem 4.8.

Assume (G1), (G2) and (H3) hold, then (16) has at least one solution on I = [0, a_0] ⊂ R, $a_0 > 0$.

Consider now the equations

$$\partial_{tgH}u = F(t, u, \partial_{xgH}u) \quad (17)$$

and

$$u(x, 0) = u_0(x) \in E^1 \quad (18)$$

Assume that :

(F1) F is locally Lipschitz continuous with respect to the second and the third variables. That is, there exists $L > 0$ such that for all u_1, $u_2 \in V$ and $V \subset E^1$ an open set

$$\|F(t,\, u_1, \partial_x u_1) \ominus_{gH} F(t,\, u_2,\, \partial_x u_2)\|_1 \\ \leq L(\|u_1 \ominus_{gH} u_2\|_1 + \|\partial_x u_1 \ominus_{gH} \partial_x u_2\|_1).$$

(F1) There exist $C_1 > 0$, $C_2 > 0$ and $M > 0$ reals such that

$$||F(t, u, \partial_x u_1)||_1 \leq C_1||u||_1 + C_2||\partial_x u||_1 + M$$

Using Theorem 3.7, we have:

Corollary 4.9.

Assume (F1), (F2) and (H3) hold, then the initial value problem (17)-(18) has at least one solution on [0, a_0] ⊂ R

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