Stock Market Indexes: A random walk test with ARCH (q) disturbances

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ABSTRACT: We will here study the stock market indexes, in the context of a random walk test with ARCH (q) disturbances. This model based on these theoretical predictions has been evaluated from the Tunis Stock market data. The coherence of the parameters signs and the statistical relevance of the estimations are validating the choice of the conditionally heteroskedastic random walk model.

KEYWORDS: white noise, index, random walk, ARCH (or GARCH) model.

1 INTRODUCTION

We will, here, study the series of the stock market quotations. The choice of the topic is dictated by considerations originating, a priori, in the economic theory and the availability of the series of observations.

2 THE CONDITIONALLY HETEROSKEDASTIC RANDOM WALK MODEL

We will, here, achieve a random walk test. We will assume that the data-generating process is a random walk process, that is to say:

\[ y_t = y_{t-1} + e_t \]

The disturbances \( e_t \) are checking the following hypotheses:

\[
\begin{align*}
E[e_t] &= 0 \\
E[e_t e_{t-s}] &= \begin{cases} 
\sigma_s^2 & \text{if } s = 0 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

The first interpretation of the random walk hypothesis is dictated by the conditional expectation, that is to say:

\[
E\left[ \frac{y_t}{y_{t-1}} \right] = E\left[ \frac{y_{t-1} + e_t}{y_{t-1}} \right] = y_{t-1}
\]

Then, the best prediction of the stock market quotations corresponds to its past value.

The choice of a random walk process means that in average, the change of quotation is zero.

This second interpretation is described in terms of the conditional expectation of the increases of the series:
Stock Market Indexes: A random walk test with ARCH (q) disturbances

\[
E \left[ \frac{y_t - y_{t-1}}{y_{t-1}} \right] = E \left[ \frac{e_t}{y_{t-1}} \right] = 0
\]

We will achieve, as a first step, the conditionally heteroskedastic random walk test:

\[
\begin{align*}
y_t &= y_{t-1} + e_t \\
e_t^2 &= c_0 + \sum_{i=1}^{q} c_i e_{t-i}^2 + \omega_t
\end{align*}
\]

Then, as \( y_t^* = y_t - y_{t-1} \)

\( = e_t \):

We will, here, study the data-generating process as a conditionally heteroskedastic white noise model:

\[
\begin{align*}
y_t^* &= e_t \\
e_t^2 &= c_0 + \sum_{i=1}^{q} c_i e_{t-i}^2 + \omega_t
\end{align*}
\]

We will, here, achieve the white noise test in the series studied:

\( y_t^* = y_t - y_{t-1} \)

\( = e_t \)

Then, \( E[y_t^*] = E[e_t] \)

\( = 0 \)

\( E[y_t^*, y_{t-s}] = E[e_t e_{t-s}] \)

\( = \gamma(s) \)

\[
\gamma(s) = E[y_t^*, y_{t-s}] = \begin{cases} 
\gamma(0) & \text{if } s = 0 \\
0 & \text{otherwise} 
\end{cases}
\]

Then:

\[
\rho_s = \frac{\gamma(s)}{\gamma(0)} = \begin{cases} 
1 & \text{if } s = 0 \\
0 & \text{otherwise} 
\end{cases}
\]

\( y_t^* \) is a white noise if \( \rho_s = 0, \ s = 1,2,... \)

Then, we will achieve a white noise test in the series \( y_t^* : \)

\[
H_0^i: \rho_i = 0 \\
H_a^i: \rho_i \neq 0
\]

\( i = 1,2,... \)
Under $H_0^i: \rho_i = 0$, the data-generating process is a white noise.

This propriety of the process studied enables us to get interested in the future values of the stock market index series:

$$E\left[\frac{Y_{i+1}}{Y_i}\right] = Y_i$$

As the matter of fact, the prediction to the time $t$ of the process studied corresponds to the current value.

The statistics associated to this test are then:

$$t_{\hat{\rho}_i} = \frac{\hat{\rho}_i - \rho_i}{\hat{\sigma}(\hat{\rho}_i)}$$

$$= T^{1/2} [\hat{\rho}_i - \rho_i]$$

Under $H_0^i: \rho_i = 0$, $T^{1/2} \hat{\rho}_i = N(0, 1)

$$\hat{\rho}_i = \hat{\gamma}(i) / \hat{\gamma}(0)$$

$$\hat{\gamma}(i) = \frac{1}{T - i} \sum_{t=i}^{T} \hat{\varepsilon}_t \hat{\varepsilon}_{t-i}.$$ 

We will reject $H_0^i: \rho_i = 0$ if $|t_{\hat{\rho}_i}| > t_{\alpha/2}$.

If $\alpha = 0.05$, $t_{0.025} = 1.96$.

The hypothesis $H_0^i: \rho_i = 0$ validates the white noise hypothesis.

We will, here, achieve a second white noise test:

$H_0: \rho_0 = \ldots = \rho_s = 0$

$H_a = H_0^c$

The alternative hypothesis indicates the complementary of $H_0$.

The statistic associated to this test is then:

$$Q = T \sum_{i=1}^{s} \hat{\rho}_i^2.$$ 

Under $H_0: \rho_0 = \ldots = \rho_s = 0$, $Q \approx \chi^2_{s}.$

We will determine the value of the Ljung –Box statistic, $Q^*$:

$$Q^* = T(T + 2) \sum_{i=1}^{s} \frac{1}{T - i} \hat{\rho}_i^2.$$ 

$Q^*$ is following a Khi –square distribution with $s$ degrees of freedom.
3 PROPERTIES OF THE SERIES STUDIED-TESTS

3.1 PROPERTIES OF THE SERIES STUDIED

The series of the stock market quotations is subject to an instantaneous variability.

We will study the series available, \( p_t \), over the period, [1997, 2008].

The data constitute the daily frequencies.

i. Inference statistics

We will, at a first step, determine the moments of order one and two, the minimum and the maximum, the skewness \( s \) and kurtosis \( k \) parameters ...

<table>
<thead>
<tr>
<th></th>
<th>( p_t )</th>
<th>( y_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>1146,25</td>
<td>0,0005</td>
</tr>
<tr>
<td>Median</td>
<td>1031,85</td>
<td>0,0002</td>
</tr>
<tr>
<td>Minimum</td>
<td>449,64</td>
<td>-0,05</td>
</tr>
<tr>
<td>Maximum</td>
<td>2346,11</td>
<td>0,04</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>473,10</td>
<td>0,0077</td>
</tr>
<tr>
<td>( s )</td>
<td>0,64</td>
<td>0,08</td>
</tr>
<tr>
<td>( k )</td>
<td>2,64</td>
<td>6,24</td>
</tr>
<tr>
<td>( N )</td>
<td>2742</td>
<td>2742</td>
</tr>
</tbody>
</table>

We will study, as a second step, in terms of rate increase, the stock market series:

\[
p_t^* = \log \left( \frac{p_t}{p_{t-1}} \right).
\]

We will compare the skewness \( s \) and the kurtosis \( k \) coefficients to the parameters values associated to a gaussian process: \( s = 0 \) and \( k = 3 \).

The values of these coefficients, corresponding to \( s = 0,08 \) and \( k = 6,24 \), do not belong to the reference values of a gaussian process.

As a matter of fact, the series of the stock market quotations is not following a normal distribution.

The value of kurtosis, which is quite high, expresses the leptokurtic aspect of the series studied.

The coefficient \( s \) does not belong to the proximity of zero. This hypothesis is significant of non-linearity.

ii. The graphic study:

The series of the stock market quotations, \( p_t \), is not, a priori, stationary (Graph a).

In the other hand, the series \( p_t^* \) is stationary (Graph b).

The graphic representation of this series is expressing the regroupings of volatility:

The values of this series, with unpredictable signs are followed by values with the same signs.

The variability of the series persists in time. These predictions are validating, a priori, of an ARCH model hypothesis, representative of a data-generating process.
3.2 Tests

We will, here, achieve a whole range of tests: a stationarity test, a normality test,

i. Stationarity test

\[
\begin{array}{c|c|c}
 p_t & ADF & p - p \\
\hline
 I(1) & I(1) &  \\
\end{array}
\]

The series of the stock market quotations \(p_t\) is not stationary.

The series \(p_t^*\) is stationary. Then, the series \(p_t\) is \(I(1)\).

ii. Normality test

Test 1: Jarque –Béra test:

We will, here, achieve the test: \(H_0^*: \begin{cases} s = 0 \\ k - 3 = 0 \end{cases}\)

\(H_a = H_0^c\).

The statistics associated to this test are then:

\[
J - B = \frac{T}{6} \left[ s^2 + \frac{1}{4} (k - 3)^2 \right].
\]

Under \(H_0^*: \begin{cases} s = 0 \\ k - 3 = 0 \end{cases}\) : \(JB \approx \chi^2(2)\).

\(s\) and \(k\) indicate the skewness and the kurtosis coefficients.

\[
\begin{array}{c|c|c}
p_t^* & \text{Statistics of the test} & p \\
\hline
 & 1197.01 & 0.000000 \\
\end{array}
\]
As \( p \leq 0.05 \), we will, here, reject:

\[
H_0 : \left\{ \begin{array}{ll}
s = k - 3 \\
= 0
\end{array} \right.
\]

The series \( y_t \) is not following a normal distribution.

**Test 2 : Granger –Newbold test**

We will, afterwards, achieve the normality test of Granger –Newbold.

We will determine the autocorrelation coefficients of order \( j \), \( \rho_j \):

<table>
<thead>
<tr>
<th>( j )</th>
<th>( \rho_j(y_t) )</th>
<th>( \rho_j(y_t^2) )</th>
<th>( \frac{\rho_j^2(y_t)}{\rho_j(y_t^2)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.313</td>
<td>0.494</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>0.111</td>
<td>0.340</td>
<td>0.04</td>
</tr>
<tr>
<td>3</td>
<td>(-0.041)</td>
<td>0.215</td>
<td>0.0080</td>
</tr>
<tr>
<td>4</td>
<td>(-0.024)</td>
<td>0.199</td>
<td>0.0028</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>23</td>
<td>(-0.015)</td>
<td>0.024</td>
<td>0.01</td>
</tr>
<tr>
<td>24</td>
<td>0.031</td>
<td>0.034</td>
<td>0.0282</td>
</tr>
</tbody>
</table>

The ratio between \( \rho_j^2(y_t) \) and \( \rho_j(y_t^2) \) does not belong to the proximity of one the series studied is not following a normal distribution. It is constitutes the general characteristic of the financial series.

It is agreed to underline that: \( y_t = \rho_t^* \)

### 4 ESTIMATION

In the choice of model, we realize a whole class of tests.

#### 4.1 TESTS ON THE MODEL

\( y_t = b_0 + e_t \)

**i. White noise test**

We will, here, achieve, on the series \( \hat{e}_t^2 \), the test:

\[
H_0 : \rho_1 = 0 \quad \text{(or } H_0 : \rho_1 = \ldots = \rho_s = 0) \]

\[
H_0^* = H_0^* .
\]

We will reject, \( H_0 : \rho_1 = 0 \) (and \( H_0 : \rho_1 = \rho_2 = 0 \)). Then, \( \hat{e}_t^2 \) is, a priori, following a second order autoregressive process: The disturbances \( e_t \) are following an ARCH (2) (or GARCH (1, 1)) model.

**ii. Test**
\[ H_0 : c_1 = \ldots = c_q = 0 \]
\[ Ha = H_0^c \]

We will achieve, in the context of this hypothesis, the regression of \( \hat{e}_t^2 \) on the values \( \hat{e}_{t-i}^2 \):

\[ \hat{e}_t^2 = c_0 + \sum_{i=1}^{q} c_i \hat{e}_{t-i}^2 + \omega_t. \]

<table>
<thead>
<tr>
<th>ARCH((q))</th>
<th>(c_0)</th>
<th>(c_1)</th>
<th>(c_2)</th>
<th>(F)</th>
<th>(NR^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARCH(1)</td>
<td>3.05 \times 10^{-5} (12,30)</td>
<td>0.48 (29,25)</td>
<td></td>
<td>255.56</td>
<td>652.34</td>
</tr>
<tr>
<td>ARCH(2)</td>
<td>2.62 \times 10^{-5} (10,38)</td>
<td>0.42 (22,15)</td>
<td>0.14 (7,44)</td>
<td>463.71</td>
<td>693.40</td>
</tr>
</tbody>
</table>

The numbers in brackets are the \(t\) of Student.

<table>
<thead>
<tr>
<th>ARCH((q))</th>
<th>(\overline{R}^2)</th>
<th>(LogL)</th>
<th>(AIC)</th>
<th>(SIC)</th>
<th>(HQC)</th>
<th>(DW)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARCH(1)</td>
<td>0.24</td>
<td>20872.52</td>
<td>-15.24</td>
<td>-15.22</td>
<td>-15.24</td>
<td>2.14</td>
</tr>
<tr>
<td>ARCH(2)</td>
<td>0.25</td>
<td>20891.86</td>
<td>-15.25</td>
<td>-15.25</td>
<td>-15.25</td>
<td>2.00</td>
</tr>
</tbody>
</table>

The values of \(LogL\) and \(\overline{R}^2\) and the information criteria validate the hypothesis of an \(ARCH(2)\) model.

We will, then, achieve the test:
\[ H_0 : c_2 = 0 \]
\[ Ha : c_2 \neq 0 \]

We will, here, determine \(SCR_0\) :

\[ SCR_0 = 3.88 \times 10^{-5} \]
\[ SCR_\alpha = 3.80 \times 10^{-5} \]

The statistics associated to this test is then:

\[ F = \frac{(SCR_0 - SCR_\alpha)}{\sqrt{\frac{1}{N-3}}} \]

This statistics is following a Fisher distribution with 1 and \(N-3\) degrees of freedom.

\[ F = \frac{(3.88 - 3.80) \times 10^{-5}}{3.80 \times 10^{-5}} (N-3) = 57.66 \]

\[ F > F_\alpha (1, N-3) : \text{We will reject } H_0 : c_2 = 0. \]
The test validates an $ARCH(2)$ process.

At the first time, the choice of model validates the following estimations:

<table>
<thead>
<tr>
<th>$\hat{b}_0$</th>
<th>$t_{\hat{b}_0}$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,000534</td>
<td>3,62</td>
<td>0,0003</td>
</tr>
</tbody>
</table>

$SCR = 0,163294$.

$LogL(y; b_0, \sigma^2) = 9443,30$

$DW = 1,37$

$AIC = -6,8896$

$SIC = -6,8875$

$HQC = -6,8888$

Nous réalisons le test :

$H_0 : b_0 = 0$

$H_a : b_0 \neq 0$

Under $H_0 : b_0 = 0$, $t_{\hat{b}_0} = \frac{\hat{b}_0}{\sigma(\hat{b}_0)} = 3,62$

as $p < 0,05$, we reject $H_0 : b_0 = 0$.

Dans un second temps, nous réalisons le test d’un processus $ARCH(q)$ :

\[
\begin{align*}
  y_t &= b_0 + \epsilon_t \\
  \epsilon_t^2 &= c_0 + c_1 \epsilon_{t-1}^2 + \omega_t
\end{align*}
\]

<table>
<thead>
<tr>
<th>$\hat{b}_0$</th>
<th>$0,000184$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1,62)</td>
<td>0,1034</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$c_0$</th>
<th>$2,5 \times 10^{-5}$</th>
<th>$0,0000$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(36,14)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$c_1$</th>
<th>0,62</th>
<th>0,0000</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(15,90)</td>
<td></td>
</tr>
</tbody>
</table>

The parameter $b_0$ is not significantly different from zero.

### 4.2 Random walk with disturbances GARCH(p, q)-Tests
We will study the statistical pertinence of estimations of the two classes of models:

i- Random walk model with disturbances GARCH (p, q),

ii- Random walk model with disturbances TGARCH (p, q).

The estimations of these models are the following:

**ARCH (1) model:**

\[
y_t = y_{t-1} + e_t \\
\hat{\epsilon}_t^2 = c_0 + c_1 \hat{\epsilon}_{t-1}^2 + \omega_t \\
\hat{\epsilon}_t^2 = 4.8 \times 10^{-5} + 0.44 \hat{\epsilon}_{t-1}^2 \\
(52,10) \quad (15,60)
\]

**ARCH (2) model:**

\[
y_t = y_{t-1} + e_t \\
\hat{\epsilon}_t^2 = c_0 + c_1 \hat{\epsilon}_{t-1}^2 + c_2 \hat{\epsilon}_{t-2}^2 + \omega_t \\
\hat{\epsilon}_t^2 = 3.8 \times 10^{-5} + 0.41 \hat{\epsilon}_{t-1}^2 + 0.20 \hat{\epsilon}_{t-2}^2 \\
(38,85) \quad (14,45) \quad (10,84)
\]

**ARCH (6) model:**

\[
y_t = y_{t-1} + e_t \\
\hat{\epsilon}_t^2 = c_0 + \sum_{i=1}^{6} c_i \hat{\epsilon}_{t-i}^2 + \omega_t
\]

<table>
<thead>
<tr>
<th>\hat{c}_i \quad</th>
<th>\hat{t}_i \quad</th>
</tr>
</thead>
<tbody>
<tr>
<td>\hat{c}_0 \quad</td>
<td>1.8 \times 10^{-5}</td>
</tr>
<tr>
<td>\hat{c}_1 \quad</td>
<td>0.41</td>
</tr>
<tr>
<td>\hat{c}_2 \quad</td>
<td>0.20</td>
</tr>
<tr>
<td>\hat{c}_3 \quad</td>
<td>0.08</td>
</tr>
<tr>
<td>\hat{c}_4 \quad</td>
<td>0.14</td>
</tr>
<tr>
<td>\hat{c}_5 \quad</td>
<td>0.10</td>
</tr>
<tr>
<td>\hat{c}_6 \quad</td>
<td>0.04</td>
</tr>
</tbody>
</table>

**GARCH (p, q) model:**

\[
y_t = y_{t-1} + e_t \\
\hat{\epsilon}_t^2 = c_0 + c_1 \hat{\epsilon}_{t-1}^2 + a_1 h_{t-1} + \omega_t
\]

<table>
<thead>
<tr>
<th>\hat{c}_0 \quad</th>
<th>\hat{c}_1 \quad</th>
<th>\hat{a}_1</th>
</tr>
</thead>
</table>

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The series of stock market quotations is following a random walk process. It is a random walk model with disturbances ARCH (or TARCH).
The value of the likelihood function and the information criteria validate the random walk hypothesis with disturbances $\text{TGARCH}(1,1)$.

5 CONCLUSION

The proprieties of the series studied are conditioning the inference methods:
- The series of stock market quotations, as any financial series, is subject to an instantaneous variability. The regrouping of volatility is described by the values of the series with unpredictable signs, which are followed by values with the same signs.
- The skewness ($s$) and the kurtosis ($k$) coefficients do not validate the hypothesis of a Gaussian process.
- In the tests on the residues, we will accept the hypothesis of an autoregressive conditionally heteroskedastic process.
- The choice of the ARCH model is dictated by the proprieties of the series studied.

REFERENCES