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Kählerian structure associated to de Sitter group

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ABSTRACT: In this paper, we consider the de Sitter algebra and we realize the Kählerian structure by using the fact that the dual of a Lie algebra of a Lie group has a natural Poisson structure and also the fact that a non-degenerate Killing form on a Lie algebra induces a metric on its dual.

KEYWORDS: Lie algebra of Lie group, Kirillov form, Killing form, Poisson bracket, Riemann bracket, Hermitian metric, Kähler manifold.

1 INTRODUCTION

The de Sitter space V_+ with positive curvature k is described by the hyperboloid

$$-u_1^{\ 2} + u_2^{\ 2} + u_3^{\ 2} = k^{-2} \tag{1}$$

where u_1 , u_2 and u_3 are global coordinates such that :

$u_1 = q$, $u_2 = k^{-1} (1 + k^2 q^2)^{1/2} \sin kct$, $u_3 = k^{-1} (1 + k^2 q^2)^{1/2} \cos kct$.

The de Sitter group dS_+ is the group of orthogonal symetries of V_+ , its acts transitively on the spaces V_+ as follows $u'_i = g_i^j u_j$. $\boldsymbol{dS_+}$ stand for the Lie algebra of Lie group dS_+ . We assume that $\boldsymbol{dS_+}$ is generated by the basis (P, K, H) where P is the infinitesimal generator of spatial translation, K is the infinitesimal generator of boost and H is the infinitesimal generator of time translation. We associate to P, K and H respectively the parameters of length (x), velocity (v) and time (t).

For the de Sitter Lie algebra $\boldsymbol{dS_+}$ endowed with the basis (P, K, H), the nontrivial Lie brackets are :

$$[P, H] = \omega^2 K, \quad [K, P] = \frac{1}{C^2} H, \quad [K, H] = P. \tag{2}$$

For more details about the de Sitter group and its Lie algebra, see [2] and [5].

We name $\boldsymbol{dS_+}$******* the dual of the Lie algebra $\boldsymbol{dS_+}$ and we recall that in the dual Lie algebra, the dual of P, K and H are linear momenta (p), static momenta (k) and energy (E).

2 POISSON STRUCTURE ASSOCIATED TO DE SITTER LIE ALGEBRA

The derivative of the coadjoint action of the de Sitter group on ***dS₊**** designed by ad*, permits to define the Kirillov form on ***dS₊****,

$$\langle ad^*_X(\alpha), Y\rangle = \langle \alpha, [X, Y]\rangle = K_{ij}(\alpha) X^i Y^j \quad (3)$$

where $\alpha(p, k, E) \in \boldsymbol{dS_+}^*$, $X, Y \in \boldsymbol{dS_+}$ and <,> is a pairing between ***dS₊*** and ***dS₊****.

In the basis (P, K, H) of ***dS₊***, the Kirillov form is given by the matrix

$$K_{ij}(\alpha) = -\alpha_k C_{ij}^{\;k}$$

where $C_{ij}^{\;k}$ are structure constants. Explicitly, the matrix of the Kirillov form in this basis of ***dS₊*** is

$$K_{ij}(p, k, E) = \begin{pmatrix} 0 & \frac{-1}{C^2}E & -\omega^2 k \\ \frac{1}{C^2}E & 0 & -p \\ \omega^2 k & p & 0 \end{pmatrix}. \quad (4)$$

For any f, g $\in C^\infty(\boldsymbol{dS_+}, \mathbb{R})$, the Poisson bracket is given by

$$\{f, g\} = K_{ij}(\alpha)\frac{\partial f}{\partial \alpha_i}\frac{\partial g}{\partial \alpha_j}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial k} & \frac{\partial f}{\partial E} \end{pmatrix} \begin{pmatrix} 0 & \frac{-1}{C^2}E & -\omega^2 k \\ \frac{1}{C^2}E & 0 & -p \\ \omega^2 k & p & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial g}{\partial p} \\ \frac{\partial g}{\partial k} \\ \frac{\partial g}{\partial E} \end{pmatrix}$$

$$= \frac{1}{C^2}E\frac{\partial f}{\partial k}\frac{\partial g}{\partial p} + \omega^2 k\frac{\partial f}{\partial E}\frac{\partial g}{\partial p} - \frac{1}{C^2}E\frac{\partial f}{\partial p}\frac{\partial g}{\partial k} + p\frac{\partial f}{\partial E}\frac{\partial g}{\partial k} - \omega^2 k\frac{\partial f}{\partial p}\frac{\partial g}{\partial E} - p\frac{\partial f}{\partial k}\frac{\partial g}{\partial E} \quad (5)$$

This Poisson bracket provides Poisson structure associated to the de Sitter group.

(***dS₊****, {, }), the dual of ***dS₊*** endowed with the Poisson bracket {,}, is a Poisson manifold.

3 RIEMANN STRUCTURE ASSOCIATED TO DE SITTER LIE ALGEBRA

We recall that the adjoint representation of ***dS₊*** is defined by

$$ad : \boldsymbol{dS_+} \to End(\boldsymbol{dS_+}), \qquad X \to adX,$$

for any $Y \in \boldsymbol{dS_+}$, $ad_X : \boldsymbol{dS_+} \to \boldsymbol{dS_+}$, $Y \to ad_X(Y) = [X, Y]$.

In the basis (P, K, H) of ***dS₊***, the matrix culumns of the endomorphism $ad_{pP+kK+EH}$ of ***dS₊*** are provided by

$$ad_{pP+kK+EH}(P) = p.ad_P(P) + k.ad_K(P) + E.ad_H(P)$$
$$= p[P,P] + k[K,P] + E[H,P]$$
$$= \frac{1}{C^2}kH - E\omega^2 K$$
$$ad_{pP+kK+EH}(K) = -\frac{1}{C^2}pH - EP$$
$$ad_{pP+kK+EH}(H) = p\omega^2 K + kP. \quad (6)$$

We get the matrix of $ad_{pP+kK+EH} \in End(\boldsymbol{dS_+})$

$$\begin{pmatrix} 0 & -E & k \\ -E\omega^2 & 0 & p\omega^2 \\ \frac{1}{C^2}k & -\frac{1}{C^2}p & 0 \end{pmatrix}. \quad (7)$$

The Killing form R is defined as follows

$$R : \boldsymbol{dS_+} \times \boldsymbol{dS_+} \to \mathbb{R}$$

$$(X,Y) \to R(X,Y) = Tr(ad_X \cdot ad_Y) \tag{8}$$

The Killing form defines a metric on the de Sitter Lie algebra ***dS₊***. Since the Killing form is non-degenerate, it defines also a metric on ***dS₊***.

From (8), for a fixed basis (X_i) of ***dS₊*** , one has

$$\begin{aligned} R_{ij} &= R(X_i, X_j) \\ &= Tr(ad_{X_i} \cdot ad_{X_j}) \end{aligned} \tag{9}$$

and the Riemann bracket on ***dS₊**** can be defined for any f, g ∈ ***dS₊**** by

$$(f,g) = R^{ij}(\alpha)\frac{\partial f}{\partial \alpha_i}\frac{\partial g}{\partial \alpha_j} \tag{10}$$

Consider now the matrix obtained in (7), for any $X = pP + kK + EH$

$X' = P'P + k'K + E'H \in$ ***dS₊***,

$$R(X,Y) = Tr\begin{pmatrix} 0 & -E & k \\ -E\omega^2 & 0 & p\omega^2 \\ \frac{1}{c^2}k & -\frac{1}{c^2}p & 0 \end{pmatrix}\begin{pmatrix} 0 & -E' & k' \\ -E'\omega^2 & 0 & p'\omega^2 \\ \frac{1}{c^2}k' & -\frac{1}{c^2}p' & 0 \end{pmatrix}$$

$$= -2\omega^2 EE' + \frac{2}{c^2}kk' - \frac{2\omega^2}{c^2}pp'$$

$$= (p\ \ k\ \ E)\begin{pmatrix} \frac{-2\omega^2}{c^2} & 0 & 0 \\ 0 & \frac{-2\omega^2}{c^2} & 0 \\ 0 & 0 & -2\omega^2 \end{pmatrix}\begin{pmatrix} p' \\ k' \\ E' \end{pmatrix} \tag{11}$$

The inverse of the matrix R_{ij} appeared in the relation (11) is

$$R^{ij} = \begin{pmatrix} \frac{-c^2}{2\omega^2} & 0 & 0 \\ 0 & \frac{c^2}{2} & 0 \\ 0 & 0 & \frac{-1}{2\omega^2} \end{pmatrix} \tag{12}$$

and allows us to defined the Riemann bracket on ***dS₊***

$$\begin{aligned} (f,g) &= R^{ij}(\alpha)\frac{\partial f}{\partial \alpha_i}\frac{\partial g}{\partial \alpha_j} \\ &= \begin{pmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial k} & \frac{\partial f}{\partial E} \end{pmatrix}\begin{pmatrix} \frac{-c^2}{2\omega^2} & 0 & 0 \\ 0 & \frac{c^2}{2} & 0 \\ 0 & 0 & \frac{-1}{2\omega^2} \end{pmatrix}\begin{pmatrix} \frac{\partial g}{\partial p} \\ \frac{\partial g}{\partial k} \\ \frac{\partial g}{\partial E} \end{pmatrix} \\ &= \frac{-c^2}{2\omega^2}\frac{\partial f}{\partial p}\frac{\partial g}{\partial p} + \frac{c^2}{2}\frac{\partial f}{\partial k}\frac{\partial g}{\partial k} - \frac{1}{2\omega^2}\frac{\partial f}{\partial E}\frac{\partial g}{\partial E} \end{aligned} \tag{13}$$

for any f, g ∈ ***dS₊*** .

(***dS₊****, (,)), the dual of ***dS₊*** endowed with the Riemann bracket (,), is a Riemann manifold, [4].

4 KAHLER STRUCTURE ASSOCIATED TO DE SITTER GROUP

The Poisson bracket {,} and the Riemann bracket (,) given respectively in (5) and (13) provide the hermitian metric, denoted by ⟨, ⟩ , and called Kahler bracket, on ***dS₊****.

For any f, g ∈ ***dS₊****,

$$\langle f, g \rangle = (f, g) + i\,\{f, g\} \tag{14}$$

With $i \in \mathbb{C}$ such that $i^2 = -1$, [3]. The Kahler bracket defines a Kahlerian structure on the dual of the de Sitter algebra. (dS_+^*, $\langle , \rangle$), the dual of dS_+ endowed with the Kahler bracket $\langle , \rangle$, is a Kahler manifold.

5 CONCLUSION

The Kähler structure plays an important role in the geometrical description of Schrödinger quantum mechanics. By the way of the Poisson bracket and the Riemann bracket, we obtain the Kähler bracket. In this paper, we want to make clear the steps leading to obtain the Kähler structure from a Lie group.

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