The Effect of the Boundary Conditions on the Onset of Convection in a Newtonian Nanofluid Layer in Presence of an Internal Heat Source: A Revised Model

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ABSTRACT: In this paper, we use a more realistic model which incorporates the effects of Brownian motion and thermophoresis of nanoparticles for studying the effect of boundary conditions and some control parameters on the onset of convective instability in presence of a uniform heat source in a confined medium filled of a Newtonian nanofluid layer which is heated uniformly from below, this layer is assumed to have a low concentration of nanoparticles. The linear study which was achieved in this investigation shows that the thermal stability of Newtonian nanofluids depends of the state of the horizontal boundaries (rigid or free), the heat source strength, the buoyancy forces, the Brownian motion, the thermophoresis of nanoparticles and other thermo-physical properties of nanoparticles. The governing differential equations are transformed into a set of ordinary differential equations by using similarity transformations, these equations will be solved analytically by converting our boundary value problem to an initial value problem, after this step we will approach the searched solutions numerically with polynomials of high degree to obtain a fifth order accurate solution.

KEYWORDS: Linear stability, Newtonian nanofluid, Heat source, Brownian motion, Thermophoresis, Power series.

1 INTRODUCTION

The nanofluid is considered as a homogeneous fluid containing colloidal suspensions of nano-sized particles named nanoparticles in the base fluid (water, ethylene glycol, oil). The nanoparticles used in nanofluids are generally prepared of metals, oxides, carbides, or carbon nanotubes. The purpose of using nanofluids is to obtain a higher values of heat transfer coefficient compared with that of the base fluid, this remarkable property make them potentially useful in many practical applications, for example in modern science and engineering including rotating machineries like nuclear reactors, petroleum industry, biochemical and geophysical problems.

The nanofluid term was introduced by Choi [1] in 1995 and remains usually used to characterize this type of colloidal suspension. Buongiorno [2] was the first researcher who treated the convective transport problem in nanofluids, he was established the conservation equations of a non-homogeneous equilibrium model of nanofluids for mass, momentum and heat transport. The thermal problem of instability in nanofluids with rigid-free and free-free boundaries was studied by Tzou [3, 4] using the eigenfunction expansions method. The onset of convection in a horizontal nanofluid layer of finite depth was studied by Nield and Kuznetsov [5]. The problem of natural convection in a confined medium filled of a Newtonian nanofluid layer has been studied in different situations by several authors [3-8], when the volumetric fraction of nanoparticles is constant at the horizontal walls limiting the layer, they found that the critical Rayleigh number can be decreased or increased by a significant quantity depending on the relative distribution of nanoparticles between the top and bottom walls.

Today, the problem of natural convection for the nanofluids is studied by some authors [9-12] using new boundary conditions for the nanoparticles which combine the contribution of the Brownian motion and the thermophoresis of nanoparticles instead to impose a nanoparticle volume fraction at the boundaries of the layer. The new model of boundary conditions assumes that the nanoparticle flux must be zero on the impermeable boundaries. D.A. Nield and A.V. Kuznetsov [9] are considered as the first ones who were used this type of boundary conditions for the nanoparticles. Until now, the precedent boundary conditions are generally used to study the problem of natural convection in nanofluids using the
Galerkin weighted residuals method (GWRM) without considering the different types of boundary conditions (free-free, rigid-free, free-rigid and rigid-rigid cases).

Our work consists of studying the Rayleigh-Bénard problem in a confined medium filled of a Newtonian nanofluid layer in the free-free, rigid-free, free-rigid and rigid-rigid cases where the nanoparticle flux is assumed to be zero on the horizontal boundaries, the studied problem will be solved with a more accurate numerical method based on analytic approximations using the power series method (PSM).

In this investigation we assume that the parameters which appear in the governing equations are considered constant in the vicinity of the temperature of the cold wall $T_c$ which we took it as a reference temperature. Finally we will impose that the flow is laminar and the radiation heat transfer mode between the horizontal walls will be negligible compared to other modes of heat transfer.

The used method gives results with an accuracy of five digits after the comma to the critical values characterizing the onset of the convection. To show the accuracy of our method in this study, we will check some results treated by Dhananjay Yadav et al. [13], Nanjundappa et al. [14] and Shivakumara and Suma [15] concerning the study of the thermal instability of regular fluids in presence of a uniform heat source.

2 MATHEMATICAL FORMULATION

We consider an infinite horizontal layer of an incompressible Newtonian nanofluid characterized by a low concentration of nanoparticles, heated uniformly from below and confined between two identical horizontal surfaces where the temperature is constant and the nanoparticle flux is zero on the boundaries (Fig.1), this layer will be subjected to an internal heat source which will provide a constant volumetric heat $Q_s$ and also to the gravity field $\vec{g}$. The thermo-physical properties of nanofluid (viscosity, thermal conductivity, specific heat) are assumed constant in the analytical formulation except for the density variation in the momentum equation which is based on the Boussinesq approximations. The asterisks are used to distinguish the dimensional variables from the nondimensional variables (without asterisks).

Within the framework of the assumptions which were made by Buongiorno [2] and Tzou [3,4] for the Newtonian nanofluids, we can write the basic equations of conservation as follows:

\[
\ddot{\vec{V}} = 0
\]

\[
\left[ \frac{\partial \dot{\vec{V}}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right] = -\nabla p + \rho \left( \frac{1}{T_0} \left( \nabla - \frac{1}{T_0} \right) \nabla - \rho \dot{\vec{V}} \right) + \eta \nabla^2 \vec{V} - \vec{g}
\]

\[
(\rho c) \left[ \frac{\partial T}{\partial t} + (\vec{V} \cdot \nabla) T \right] = \kappa \nabla^2 T + (\rho c) \left[ D \nabla \cdot \nabla T + \frac{D T}{T_0} \right] + Q_s
\]

\[
\frac{\partial \dot{\vec{X}}}{\partial t} + (\vec{V} \cdot \nabla) \dot{\vec{X}} = D \nabla^2 \dot{\vec{X}} + \frac{D T}{T_0} \nabla^2 T
\]

Where $\vec{V}$ is the vector differential operator.
If we consider the following dimensionless variables:

\[
(x^*; y^*; z^*) = H(x; y; z) \quad ; \quad t^* = \frac{H^*}{t} \quad ; \quad \tilde{V}^* = \frac{\alpha}{H^*} \tilde{V} \quad ; \quad P^* = \frac{\eta^*}{H^*} P \quad ; \quad T^* - T_e = (T_b - T_e) T^* - \chi^* = \chi^*_0
\]

Then, we can get from equations (1)-(4) the following adimensional forms:

\[
\tilde{V} \cdot \tilde{V} = 0
\]  

\[
P_t^{-1} \left[ \frac{\partial \tilde{V}}{\partial t} + (\tilde{V} \cdot \nabla) \tilde{V} \right] = -\tilde{V} (P + R_M z) + \left( 1 - \chi^*_0 \right) \tilde{V} \cdot \nabla \tilde{V} + \tilde{V} \cdot \nabla \tilde{V} + N_a L_e^{-1} \tilde{V} \cdot \nabla \tilde{V} + H_s
\]

\[
\frac{\partial T}{\partial t} + (\tilde{V} \cdot \nabla) T = \tilde{V} \cdot \nabla T + N_a L_e^{-1} \tilde{V} \cdot \nabla T + N_A N_a L_e^{-1} \tilde{V} \cdot \nabla T + H_s
\]

\[
\frac{\partial \tilde{\chi}}{\partial t} + (\tilde{V} \cdot \nabla) \tilde{\chi} = L_e^{-1} \tilde{V} \cdot \nabla \tilde{\chi} + N_A L_e^{-1} \tilde{\chi} \nabla^2 T
\]

Such that:

\[
P_t = \frac{\eta}{\rho \alpha} \quad ; \quad L_e = \frac{\alpha}{D_a} \quad ; \quad H_s = \frac{Q H^2}{\kappa (T_b - T_e)} \quad ; \quad N_a = \frac{(pc)^{\rho}}{(pc)^{\rho + 1}} \rho_0 \tilde{\chi}^*_0 \quad ; \quad \alpha = \frac{\kappa}{(pc)} \quad ; \quad R_{M} = \frac{\rho_0 g \beta H^2 (T_b - T_e)}{\eta \alpha}
\]

\[
R_{M} = \frac{\left[ \rho_0 (1 - \chi^*_0) + \rho_2 \chi^*_0 \right]}{\eta \alpha g H^2} \quad ; \quad R_{N} = \frac{\left( \rho_0 - \rho_1 \right) \chi^*_0 g H^2}{\eta \alpha} \quad ; \quad N_A = \frac{D_a}{D_a T_b \left( \frac{T_b - T_e}{\chi^*_0} \right)}
\]

Where \( \chi^*_0 \) is the reference value for nanoparticle volume fraction.

2.1 Basic Solution

The basic solution of our problem is a quiescent thermal equilibrium state, it's assumed to be independent of time where the equilibrium variables are varying in the z-direction, therefore:

\[
\tilde{V}_b = 0
\]

\[
T_b = 1 \quad ; \quad \frac{d \chi_b}{dz} + N_A \frac{dT_b}{dz} = 0 \quad \text{at} \quad z = 0
\]

\[
T_b = 0 \quad ; \quad \frac{d \chi_b}{dz} + N_A \frac{dT_b}{dz} = 0 \quad \text{at} \quad z = 1
\]

If we introduce the precedent results into equations (6)-(8), we obtain:

\[
\tilde{V} (P_b + R_{M} z) = \left[ 1 - \chi^*_0 \right] R_{N} T_b - R_{N} \chi^*_0 - \chi^*_0 R_{N} T_e \chi_b \quad \tilde{\chi}_z
\]

\[
\frac{d^2 T_b}{dz^2} + N_A L_e^{-1} \left( \frac{d \chi_b}{dz} \frac{dT_b}{dz} \right) + N_A N_a L_e^{-1} \left( \frac{dT_b}{dz} \right)^2 = -H_s
\]

\[
\frac{d^2 \chi_b}{dz^2} + N_A \frac{d^2 T_b}{dz^2} = 0
\]

After using the boundary conditions (10) and (11), we can integrate the equation (14) between 0 and z for obtaining:

\[
\chi_b = N_A \left( 1 - T_b \right) + \chi_0
\]

Where \( \chi_0 \) is the relative nanoparticle volume fraction at \( z = 0 \), such that:

\[
\chi_0 = \frac{\chi_b(0) - \chi_0}{\chi_0}
\]

If we take into account the expression (15), we can get after simplification of the equation (13):

\[
\frac{d^2 T_b}{dz^2} = -H_s
\]
Finally, we obtain after an integrating of the equation (16) between 0 and z:

\[ T_s = -\frac{1}{2} H_s z^2 + \left( \frac{1}{2} H_s - 1 \right) z + 1 \]  

(17)

\[ \chi_s = -\frac{1}{2} N_s H_s z^2 - N_s \left( \frac{1}{2} H_s - 1 \right) z + \chi_o \]  

(18)

2.2 Stability Analysis

For analyzing the stability of the system, we superimpose infinitesimal perturbations on the basic solutions as follows:

\[ T = T_0 + T'; \quad \bar{V} = \bar{V}_s + \bar{V}'; \quad P = P_s + P'; \quad \chi = \chi_s + \chi' \]  

(19)

In the framework of the Oberbeck-Boussinesq approximations, we can neglect the terms coming from the product of the temperature and the volumetric fraction of nanoparticles in equation (6), if we suppose also that we are in the case of small temperature gradients in a dilute suspension of nanoparticles, we can obtain after introducing the expressions (19) into equations (5)-(8) the following linearized equations:

\[ \bar{V}, \bar{V}' = 0 \]  

(20)

\[ p_t^{-1} \frac{\partial \bar{V}'}{\partial t} = -\bar{V}P' + (R_s T' - R_s \chi') \bar{c}_z + \bar{V}^2 \bar{V}' \]  

(21)

\[ \frac{\partial T'}{\partial t} + f_s w' = \bar{V}^2 T' + f_s \frac{\partial T'}{\partial z} + f_s \frac{\partial \chi'}{\partial z} \]  

(22)

\[ \frac{\partial \chi'}{\partial t} + f_s w' = N_s \bar{c}_z \bar{V}^2 T' + L \bar{V}^2 \chi' \]  

(23)

Such that:

\[ f_s = DT_o; \quad f_2 = N_s b_2 \bar{c}_z D(T_o + 2 N_s T_o); \quad f_s = N_s b_2 \bar{c}_z D T_o; \quad f_4 = D \chi_o; \quad D = d/dz \]

After application of the curl operator twice to equation (21) and using the equation (20), we obtain the following equation:

\[ p_t^{-1} \frac{\partial }{\partial t} \bar{V}^2 w' = \bar{V}^2 w' + R_s \bar{V}^2 T' - R_s \bar{V}^2 \chi' \]  

(24)

Such that:

\[ \bar{V}^2 = \left( \frac{\partial^2}{\partial x^2} \right) + \left( \frac{\partial^2}{\partial y^2} \right) \]

Analyzing the disturbances into normal modes, we can simplify the equations (22)-(24) by assuming that the perturbation quantities are of the form:

\[ (w', T', \chi') = (w(z), T(z), \chi(z)) \exp i \left[ (k_x x + k_y y) + \omega t \right] \]

(25)

After introducing the expressions (25) into equations (22)-(24), we obtain:

\[ p_t^{-1} \sigma \left( D^2 \cdot k^2 \right) w = \left( D^2 - k^2 \right) w - k^2 R_s T + k^2 R_s \chi \]

(26)

\[ \sigma T + f_s w = \left( D^2 - k^2 \right) T + f_s \frac{\partial T}{\partial z} + f_s \frac{\partial \chi}{\partial z} \]

(27)

\[ \sigma \chi + f_s w = N_s L \left( D^2 - k^2 \right) T + L \left( D^2 - k^2 \right) \chi \]

(28)

Where \( k \) is the resultant dimensionless wave number, such that:

\[ k = \sqrt{k_x^2 + k_y^2} \]

The equations (26)-(28) will be solved subject to the following boundary conditions:

- For the rigid-rigid case:

\[ w = Dw = T = D(\chi + N_s T) = 0 \quad \text{at} \quad z = 0 \; ; \; 1 \]

(29)
- For the free-free case;
  \[ w = D^2w = T = D(\chi + N_\alpha T) = 0 \quad \text{at} \quad z = 0 ; \quad 1 \quad (30) \]

- For the rigid-free case;
  \[ w = Dw = T = D(\chi + N_\alpha T) = 0 \quad \text{at} \quad z = 0 \quad (31) \]
  \[ w = D^2w = T = D(\chi + N_\alpha T) = 0 \quad \text{at} \quad z = 1 \quad (32) \]

- For the free-rigid case;
  \[ w = D^2w = T = D(\chi + N_\alpha T) = 0 \quad \text{at} \quad z = 0 \quad (33) \]
  \[ w = Dw = T = D(\chi + N_\alpha T) = 0 \quad \text{at} \quad z = 1 \quad (34) \]

2.3 Method of Solution

Very recently, Nield and Kuznetsov [9] and Agarwal [12] observed that the oscillatory convection is ruled out for nanofluids with this new type of boundary conditions due to very large nanofluid Lewis number, so the stationary convection is the predominant mode. Hence, the equations (26)-(28) become:

\[
\left( D^2 - k^2 \right)^2 w - k^2 R_y T + k^2 R_y \chi = 0
\quad (35)
\]

\[ f_4 w - \left( D^2 - k^2 \right) T - f_2 D T - f_3 D \chi = 0 \quad (36) \]

\[ f_4 w - N_\alpha L_e \left( D^2 - k^2 \right) T - L_e \left( D^2 - k^2 \right) \chi = 0 \quad (37) \]

We can solve the equations (35)-(37) which are subjected to the conditions (29),(30),(31) and (32) or (33) and (34) by using a suitable change of variables that makes the number of variables equal to the number of boundary conditions, to obtain a set of eight first order ordinary differential equations which we can write it in the following form:

\[ \frac{d}{dz} u_i (z) = a_{ij} u_j (z) \quad ; \quad 1 \leq i, j \leq 8 \quad (38) \]

With:

\[ a_{ij} = a_{ij} (z, k, R_y, H_e, N_\alpha, L_e, R_y, N_\alpha) \]

The solution of the system (38) in matrix notation can be written as follows:

\[ U = BC \quad (39) \]

Where B is a square matrix of order \( 8 \times 8 \), U is the unknown vector column of our problem and C is a constant vector column, such that:

\[ B = \begin{pmatrix} \left( b_i (z) \right)_{i=1}^{8} \\ \left( b_i (z) \right)_{i=1}^{8} \end{pmatrix} \]

\[ U = \begin{pmatrix} \left( u_i (z) \right)_{i=1}^{8} \end{pmatrix}^T \]

\[ C = \begin{pmatrix} \left( c_i \right)_{i=1}^{8} \end{pmatrix}^T \]

If we assume that the matrix B is written in the following form:

\[ B = \begin{pmatrix} \left( u_i^T (z) \right)_{i=1}^{8} \left( u_i^T (z) \right)_{i=1}^{8} \end{pmatrix} \quad (40) \]
Therefore, the use of four boundary conditions at \( z = 0 \), allows us to write each variable \( u_i(z) \) as a linear combination for four functions \( u_i^j(z) \), such that:
\[
u_i^j(0) = \delta_{ij}
\]
(41)

Where \( \delta_{ij} \) is the Kronecker delta symbol.

After introducing the new expressions of the variables \( u_i(z) \) in the system (38), we will obtain the following equations:
\[
\frac{d}{dz} u_i^j(z) = a_{ij} u_i^j(z) \quad ; \quad 1 \leq i, l, j \leq 8
\]
(42)

For each value of \( j \), we must solve a set of eight first order ordinary differential equations which are subjected to the initial conditions (41), by approaching the variables \( u_i^j(z) \) with real power series defined in the interval \([0,1]\) and truncated at the order \( N \), such that:
\[
u_i^j(z) = \sum_{p=0}^{N} d_p^j z^p
\]
(43)

A linear combination of the solutions \( u_i^j(z) \) satisfying the boundary conditions (29), (30), (31) and (32) or (33) and (34) at \( z = 1 \) leads to a homogeneous algebraic system for the coefficients of the combination. A necessary condition for the existence of nontrivial solution is the vanishing of the determinant which can be formally written as:
\[
f(R_e, k, H_s, N_B, L_e, R, N) = 0
\]
(44)

If we give to each control parameter \((H_s, N_B, L_e, R, N)\) its value, we can plot the neutral curve of the stationary convection by the numerical research of the smallest real positive value of the thermal Rayleigh number \( R_e \) which corresponds to a fixed wave number \( k \) and verifies the dispersion relation (44). After that, we will find a set of points \((k, R_e)\) which help us to plot our curve and find the critical value \((k_c, R_{ec})\) which characterizes the onset of the convective stationary instability, this critical value represents the minimum value of the obtained curve.

2.4 Validation of the Method

The truncation order \( N \) which correspond to the convergence of our method is determined, when the five digits after the comma of the critical thermal Rayleigh number \( R_{ec} \) for the regular fluids and the nanofluids remain unchanged. To validate our method, we compared our results with those obtained by Dhananjay Yadav et al. [13], Nanjundappa et al. [14] and Shivakumara and Suma [15] concerning the effect of an internal heat source on the onset of convective instability for the Rayleigh-Bénard problem in the case of regular fluids. To make this careful comparison, we must take in the governing equations the following restrictions:
\[
L e^j = R_N = N_A = N_B = 0
\]

According to the results described below in Tables 1-3, we notice that there is a very good agreement between our results and the previous works, hence the accuracy of the used method. From the Tables 1-5 we show also that the convergence of the results depends greatly on the truncation order \( N \) of the power series, of the type of boundary conditions and also of the values of the heat source strength \( H_s \), such that for the large values of the heat source strength \( H_s \), it's necessary to choose the greater values of the truncation order \( N \).

Finally, to ensure the accuracy of the obtained critical values for the studied nanofluids, we will take as truncation order:
\[
N^e = 32 \quad ; \quad N^f = 32 \quad ; \quad N^r = 40 \quad ; \quad N^o = 39
\]

For the regular fluids, we take:
\[
N^e = 33 \quad ; \quad N^f = 34 \quad ; \quad N^r = 40 \quad ; \quad N^o = 40
\]

The precedent values are taken when we want to vary the values of the heat source strength \( H_s \) from 0 until 60.
Table 1. The comparison of critical values of Rayleigh number and the corresponding wave number with Dhananjay Yadav et al. [13], for the regular fluids in the free-free case.

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<th>N</th>
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Table 2. The comparison of critical values of Rayleigh number and the corresponding wave number with Nanjundappa et al. [14], for the regular fluids in the rigid-free case.

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Table 3. The comparison of critical values of Rayleigh number and the corresponding wave number with Shivakumara and Suma [15], for the regular fluids in the rigid-rigid case.

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Table 4. Our critical values of Rayleigh number and their corresponding wave numbers, for the regular fluids in the free-rigid case.

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Exact value $3.06865$ $1359.69716$ $3.72384$ $895.33580$ $3.91104$ $532.39508$

Table 5. Our critical values of Rayleigh number and their corresponding wave numbers, for a nanofluid characterized by $N_b = 0.01$, $Le = 100$, $R_e = 1$ and $N_a = 0.1$ in the case where $H_s = 60$.

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Exact value $2.80818$ $210.83243$ $2.72348$ $209.34745$ $3.79041$ $505.72723$ $3.88686$ $517.06031$

3 RESULTS AND DISCUSSION

To study the effect of a parameter $(H_s, N_B, L_c, R_e, N_A)$ on the onset of the convective instability in a confined medium filled of a Newtonian nanofluid layer, we must plot in Figs 2-5 the variation of the critical thermal Rayleigh number $R_{ac}$ as a function of the heat source strength $H_s$ for different values of this parameter and compare the obtained results with those of the regular fluids. For this purpose, we will consider a reference nanofluid characterised by $N_B = 0.01$, $L_e = 100$, $R_e = 1$, $N_A = 0.1$ and then plot the variations of the critical thermal Rayleigh number $R_{ac}$ with the heat source strength $H_s$ in the interval $[0,60]$ for different values of the modified particle-density increment $N_B$ in Fig.2, the Lewis number $L_e$ in Fig.3, the concentration Rayleigh number $R_N$ in Fig.4 and the modified diffusivity ratio $N_A$ in Fig.5.

In Figs 2-5, we plot also the variation of the critical thermal Rayleigh number as a function of the heat source strength $R_{ac} = f(H_s)$ for the regular fluids in the free-free, rigid-free, free-rigid, and rigid-rigid cases.
Fig. 2. The variation of $R_{ac}$ as a function of $H_s$ for different values of $N_B$ in the case where $L_e = 100$, $R_N = 1$ and $N_a = 0.1$
Fig. 3. The variation of $R_e$ as a function of $H_s$ for different values of $L_s$ in the free-free case (a), free-rigid case (b), rigid-free case (c) and rigid-rigid case (d) for $N_u = 0.01$, $N_A = 1$ and $N_A = 0.1$

Fig. 4. The variation of $R_e$ as a function of $H_s$ for different values of $R_N$ in the free-free case (a), free-rigid case (b), rigid-free case (c) and rigid-rigid case (d) for $N_u = 0.01$, $L_s = 100$ and $N_A = 0.1$
Fig. 5. The variation of $R_e$ as a function of $H_s$ for different values of $N_A$ in the free-free case (a), free-rigid case (b), rigid-free case (c) and rigid-rigid case (d) for $N_B = 0.01$, $L_e = 100$ and $R_n = 1$

Whatever the type of boundary conditions (free-free, free-rigid, rigid-free and rigid-rigid cases), we find graphically through the Fig.2 that there is no effect of the modified particle-density increment $N_B$ on the convective instability for the nanofluids. To determine the exact effect of the parameter $N_B$, we must determine some points of this figure (Fig.2) in the Table 6, this table shows that an increase in the modified particle-density increment $N_B$ allows us to destabilize somewhat the nanofluids, this result may be explained by its low value ($N_B \approx 10^{-3} - 10^{-1}$) which appears only in the perturbed energy equation (22) as a product with the inverse of the Lewis number ($L_e \approx 10^{3} - 10^{1}$) near the temperature gradient and the volume fraction gradient of nanoparticles, so the effect of this parameter on the onset of convection will be very small.

From the expression of the parameter $N_B$, we can conclude that the use of the nanoparticles which are characterized by a small heat capacity or a low concentration allows us to stabilize the nanofluids. In this investigation, we find that an increase in the volume fraction of nanoparticles allows us to destabilize the nanofluids, because an increase in this parameter, increases also the Brownian motion and the thermophoresis of nanoparticles, which cause a destabilizing effect. This result confirm that the regular fluids are more stable than the nanofluids (Figs 2-5).
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From Fig. 3 and Fig. 4, we conclude that an increase either in the Lewis number $L_e$ or in the concentration Rayleigh number $R_{ac}$ allows us to accelerate the onset of the convection, hence they have a destabilizing effect. Therefore, to ensure the stability of the system, we can use a nanofluid characterized by a less thermal diffusivity or constituted of less dense nanoparticles.

When the modified diffusivity ratio $N_R$ increases, the temperature difference between the horizontal plates also increases. The Fig. 5 shows that an increase in the modified diffusivity ratio $N_R$ allows us to decrease the critical thermal Rayleigh number $R_{ac}$, this result can be explained by the increase in the buoyancy forces which destabilizes the system.

Our results show the existence of the free-rigid case which is different to the rigid-free case as long as there is an internal heat source which produces a constant volumetric heat. In the free-rigid case, we find that the variation of the critical thermal Rayleigh number $R_{ac}$ as a function of the heat source strength $H_s$ presents a maximum value at $H_s = \text{max}$, but for the other cases we conclude that the precedent variation $R_{ac} = f(H_s)$ is always a decreasing function, so an increase in the heat source strength $H_s$ has a destabilizing effect on the nanofluids and the regular fluids except for the free-rigid case where the system can’t be destabilized only if the heat source strength $H_s$ exceeds a certain value $H_s = \text{max} = 8.89329$ for the reference nanofluid and $H_s = \text{max} = 8.86444$ for the regular fluids (Fig.6), such that after this value we observe that the curve which corresponds to the rigid-rigid case intersects with that of the free-rigid case at $H_s = 17.24510$ for the reference nanofluid and $H_s = 17.33104$ for the regular fluids, and then becomes its asymptotic branch. For the other cases, we find that the decrease of the curve which corresponds to the rigid-case tends asymptotically towards the curve of the free-rigid case such that the two curves can intersected at $H_s = 73.20448$ for the reference nanofluid and $H_s = 75.31258$ for the regular fluids.
In this study, we show that the presence of nanoparticles in a base fluid can return the heat source strength $H_s$ among the principal driving forces which produce the onset of convection (Table 7).

### Table 7. Some critical values for the regular fluids and the reference nanofluid ($N_a = 0.01, \Gamma_a = 100, \Gamma_n = 1, N_a = 0.1$)

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4 CONCLUSIONS

In this paper, we have examined the effect of an internal heat source on the onset of convection in a Newtonian nanofluid layer heated uniformly from below in the case where the nanoparticle flux is assumed to be zero on the horizontal boundaries. The presence of friction on the horizontal walls is a factor producing the thermal stability of the system, where the rigid-rigid case is the more stable case compared with the free-rigid, rigid-free and free-free cases, such that:

$$R^e_{ac} > R^f_{ac} > R^e_{ac} > R^f_{ac}$$

This study, shows that the free-rigid case appears only when we have an internal heat source which allows us to minimize the friction effect on the bottom wall and increase it on the top wall, such that from a certain value of the heat source strength $H_s$ we observe a reconciliation between the rigid-free and free-free cases and also between the rigid-rigid and free-rigid cases.

To ensure the stability of the system, we can use the nanoparticles which are characterized by a small heat capacity or a low concentration, we can also use the nanofluids which are having a less thermal diffusivity or constituted of less dense nanoparticles.

An increase in the volume fraction of nanoparticles, in the buoyancy forces, in the Brownian motion or in the thermophoresis of nanoparticles allows us to destabilize the nanofluids, such that the regular fluids are more stable than the nanofluids.
In this investigation we find that the presence of nanoparticles in a base fluid can return the heat source strength $H_s$ among the principal driving forces which produce the onset of convection.

The used method to solve the convection problem with the new model of boundary conditions of nanoparticles gives more accurate results, because the absolute error of the obtained critical values which characterize the onset of the convection is of the order of $10^{-9}$, Hence, we can use our results as a reference to validate other results of the similar problems.

**NOMENCLATURE**

**Symbols**:

- $c$ Specific heat of nanofluid (J/kg K)
- $D_B$ Brownian diffusion coefficient (m$^2$/s)
- $D_T$ Thermophoretic diffusion coefficient (m$^2$/s)
- $g$ Acceleration due to gravity (m/s$^2$)
- $H$ Layer depth (m)
- $H_s$ Dimensionless constant heat source strength
- $\kappa$ Thermal conductivity of Nanofluid (W/K m)
- $k_x^*$ Wave number in $x^*$ direction (m$^{-1}$)
- $k_y^*$ Wave number in $y^*$ direction (m$^{-1}$)
- $k_c^*$ Critical wave number (m$^{-1}$)
- $L_e$ Lewis number
- $N_A$ Modified diffusivity ratio
- $N_B$ Modified particle-density increment
- $P^*$ Pressure (Pa)
- $Pr$ Prandtl number
- $Q_s$ Volumetric internal heat source (J/m$^3$)
- $R_s$ Thermal Rayleigh number
- $R_{sc}$ Critical Rayleigh number
- $R_M$ Density Rayleigh number
- $R_N$ Concentration Rayleigh number
- $\vec{V}^*$ Velocity vector (m/s)
- $T^*$ Temperature (K)
- $t^*$ Time (s)
- $u^*, v^*, w^*$ Velocity components (m/s)
- $x^*, y^*, z^*$ Cartesian coordinates (m)
Greek symbols:

- $\alpha$: Thermal diffusivity of nanofluid (m$^2$/s)
- $\beta$: Thermal expansion coefficient of base fluid (K$^{-1}$)
- $\eta$: Viscosity of nanofluid (Pa.s)
- $\rho$: Nanofluid density (kg/m$^3$)
- $\rho_0$: Nanofluid density at reference temperature (kg/m$^3$)
- $\rho_c$: Heat capacity of nanofluid (J/m$^3$.K)
- $\sigma^*$: Growth rate of disturbances (s$^{-1}$)
- $\chi^*$: Volume fraction of nanoparticles

Superscripts:

- $^*$: Dimensional variable
- $'$: Perturbation variable
- $ff$: Free - Free case
- $rf$: Rigid - Free case
- $fr$: Free - Rigid case
- $rr$: Rigid - Rigid case

Subscripts:

- $c$: Cold
- $h$: Hot
- $ac$: Critical number
- $b$: Basic solution
- $f$: Base fluid
- $p$: Nanoparticle

REFERENCES


