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PROBLEM-SOLVING EFFECTS IN TEACHING AND LEARNING MATHEMATICS

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ABSTRACT: Students' performance in the area of mathematics is a topic of concern in the whole word, with several reports documenting the need of effective instruction to boost students' achievement. However, what type of math instruction will most effectively raise students' achievement remains a matter of debate. Problem-based learning is a promising methodology for engaging and motivating students' learning while increasing their math concepts and skills. Problem solving is central to mathematics. Problem solving should be the site in which all of the strands of mathematics proficiency converge. It should provide opportunities for students to weave together the strands of proficiency and for teachers to assess students' performance on all of the strands. Problem solving approach guides students through complex problems and it must remain part of day-to-day instruction because solving problems is central to doing and learning mathematics. We have provided and discussed effective teaching and learning methods from the literature as well as making suggestions and recommendations to use problem-solving methods during instruction in the classroom.

KEYWORDS: Classroom, Critical, Instruction, Knowledge, Practice, Representation, Student, Teacher.

1 INTRODUCTION

This study is aimed to present the effects of an instructional intervention (i.e., teaching mathematics through problem-solving contexts) on adolescents' problem-solving performance and representation use when solving word problems. Students' performance on a test of word problems as well as their use of different representations will be examined and compared to peers experiencing their everyday instruction from their classroom teacher. Instruction in the intervention classroom emphasizes student-to-student discourse. The intervention's intent is to enhance mathematics learning by examining, solving, and reflecting on word problems. Students need frequent opportunities to engage in problem solving so that they can become mathematically proficient. Mathematical proficiency characterizes learning mathematics successfully in such a way that one develops (a) conceptual understanding, (b) procedural fluency, (c) strategic competence, (d) adaptive reasoning, (e) and a productive disposition toward mathematics. Mathematically proficient students exhibit problem-solving behaviors such as reading problems carefully and understanding them, creating models, and making conjectures about strategies and solutions. On the other hand, students lacking mathematical proficiency demonstrate ineffective mathematical behaviors such as attempting to solve problems without making sense of the problem's context. Moreover, they are less likely to use their knowledge of mathematics content while problem solving. Mathematics practice supports mathematics learning and problem-solving performance by engaging students in daily mathematics instruction that integrates problem-solving performance by engaging students in daily mathematics instruction that integrates problem-solving features. Problem-solving goes beyond the typical thinking and reasoning students employ while solving exercises. It means thinking deeply about concepts, their associated representations, viable solution procedures, related context or cultural knowledge, and creating problem models. The literature suggests that effective problem solvers go through six stages of problem solving. A brief outline of these stages follows. First, individuals read the problem and work to understand the text. Understanding leads to a situation model, which adequately characterizes the mathematical and nonmathematical elements of the problem. It supports individuals to generate an appropriate representation of the problem, which facilitates mathematical analysis. This analysis is the same notion as implementing a set of procedures. After employing procedures, problem solvers arrive at a result, termed the derivation. Individuals interpret this derivation in light of their situation model to generate an interpreted result that might become part of the final answer and is later reported as the final answer, which completes the process. An incorrect situation model that does not reflect the problem may lead problem solvers to believe that the result is correct. Hence, creating an accurate situation model is critically important. If the interpreted result does not match the expectation of the situation model, an effective problem solver revisits the situation model and begins the process again. Many problem solvers tend to skip steps throughout this process, which often leads to incorrect results. Superficial problem solving is characterized by four steps: (1) reading the problem's text, (2) creating a mathematical model, (3) implementing a representation and set of procedures, and (4) reporting results and many students solve problems in this fashion. Problem solving as viewed from a mathematics education perspective is the process of interpreting a situation mathematically, which usually involves several iterative cycles of expressing, testing and revising mathematical interpretations and of sorting out, integrating, modifying, revising, or refining clusters of mathematical concepts from various topics within and beyond mathematics. It is a complex activity that requires individuals to maintain focus and both rationally and effectively proceed through the problem. Students' ineffective problem-solving behaviors and disengagement in the process is accelerated by teacher-directed instruction that frequently uses too many exercises and not enough problems. One way to foster students' success in the problem-solving process is to provide them with frequent opportunities to engage in problem solving in a student-centered environment that scaffolds students to successfully complete each stage of the process. Multiple studies have demonstrated that when daily mathematics instruction is integrated or supplemented with problem-solving activity, it enhances students' problem-solving capabilities. Moreover, there is some evidence that students' learning in classroom environments where problem solving is a regular part of mathematics instruction outperform their peers in traditional learning environments on mathematics achievement tests. Success on problem-solving and achievement measures is also influenced by the degree to which students are supported to gain facility with representations and procedures. Strategy (i.e., representation and procedure) use is a critical component of problem solving. Effective problem solvers actively monitor their actions while implementing a strategy. They consider a variety of procedures and representations that are suitable for completing a task and monitor their progress while completing the procedures. Instruction that allows students to consider a variety of representations and procedures to complete a task and share them has been shown to have positive effects on students' achievement. Creating an instructional context that stimulates mathematical discussions among problem solvers enhances their ability to solve problems and use a variety of representations and procedures. The teacher is the critical factor in making such a learning environment. The teacher is an important person in the classroom because this individual makes instructional decisions such as their choice of materials and instruction that influence students' mathematics learning and problem-solving performance. Teachers decide whether to enact teacher-directed instruction or foster student-centered instruction. Teacher-directed instruction is characterized by lecture stemming from the teacher's knowledge and a lack of discussion about mathematics. Students are expected to watch passively, listen to their teacher, and later practice what the teacher showed them. Discourse in teacher-directed classrooms tends to follow a three-turn interaction termed Initiate-Respond-Evaluate, which includes a teacher's inquiry students' response, and teacher's evaluative statement. Students in these teacher-centered classrooms tend to think that doing mathematics means following the rules laid down by the teacher; knowing mathematics means remembering and applying the correct rule when the teacher asks a question, and mathematical truth is determined when the answer is ratified by the teacher. On the other hand, student-centered instruction involves attending to students' knowledge and building on these ideas in a meaningful way that promotes learning about concepts and procedures. The teacher works to orchestrate the content, representations of the content, and the people in the classroom in relation to one another. Students' ways of being, their forms of participation, and their learning emerge out of these mutually constitutive relationships. The teacher in the student-centered classroom plays the role of learning facilitator and guide during mathematics instruction. Students discuss mathematics, make conjectures, and construct mathematical arguments and proofs in student-centered classrooms. Generally it [student-centered instruction] implies an approach in which learners are given opportunities to offer their own ideas and to become actively involved in their learning. During an observation of a student-centered classroom, one might notice students in small groups trading ideas and making sense of a problem, a teacher and students collaborating to solve a problem, and there are likely established social and socio mathematical norms for doing mathematics in the problem-solving oriented, student-centered mathematics classroom. Studies published in the previous two decades provide a good foundation for implementing effective mathematics instruction. It has been shown that (a) practicing procedures over and over does not develop students' mathematical understanding (b) less time spent practicing procedures does not hinder students' ability to solve routine problems and (c) spending more time on one problem in conjunction with meaningful mathematical discourse creates an opportunity for reflection and analytical thinking that facilitates students' mathematical proficiency. With this in mind, mathematics education researchers seek to support students to become effective problem solvers by implementing instructional interventions. A variety of research designs have been implemented to explore how students solve problems and to enhance their problem-solving performance and representation use. Mathematics educators suggest that conducting problem-solving research in the classroom, with students from a variety of levels, and focusing on instruction is needed to enhance researchers and practitioners' understanding of problem solving. The researchers aim to determine whether students receiving the intervention had better problem-solving performance and achievement, and whether they developed more effective problem-solving behaviors compared to their peers experiencing their typical instruction. These studies characterize supplementing mathematics instruction with problem-solving components, but their conclusions do not shed light on effects of teaching mathematics through problem-solving contexts on a daily basis. Moreover, these studies do not characterize daily mathematics instruction that teaches mathematics at high quality standards through problem-solving contexts. Prior investigations provide a foundation for examining teaching mathematics associated with the Standards through problem-solving contexts on students' problem-solving performance and representation use. Research is necessary to determine whether students' outcomes from this type of instruction differ from prior problem-solving interventions or every day instruction. One way to begin such investigations is for a researcher to become an instructor in the classroom. Some investigators set out with a goal of better understanding the teaching and learning process and they also expected to improve students' problem-solving performance and problem-solving behaviors. One group of students learned one representation and procedure at a time and completed exercises to improve their efficiency. A similar group of students experienced instruction that encouraged them to generate multiple representations and procedures to solve problems, and the teacher presented the class with more than one representation to solve these problems. Together, these studies provide a foundation for examining the effects of teaching mathematics through problem-based contexts in a student-centered, discourse-rich classroom on university students' problem-solving success and students' use of representations. The present study aims to bridge the areas of problem-solving and representation use.

2 PURPOSE OF THE STUDY

The purpose of this study was to investigate the effects of student-centered, discourse-rich mathematics instruction (i.e., instructional intervention) on students' problem-solving performance as well as their representation use when solving word problems. Creating a supportive instructional context that used word problems as the focal activity was intended to support students' opportunities for learning mathematics content and procedures. A fundamental desired outcome of the instructional intervention was to assist university students in becoming more effective and efficient problem solvers within the context of mastering mathematics content and demonstrating effective mathematical practices. A secondary desired outcome was to demonstrate that teaching mathematics from the University Standards through problem-solving contexts was possible in the midst of this critical time with new standards for mathematics content and practice.

3 STATEMENT OF THE PROBLEM

The mathematics education research and teaching community is making positive steps in helping students become better problem solvers. The Trends in Mathematics and Science Study during the last decade is promising, but there is more that can be done to support students through problem solvers' mathematical development. Although students are improving, they tend to have difficulty completing solutions of problems and little is known about problem solver's success on problems drawing on students' knowledge of out-of-classroom contexts. A problem is a task such that a path to the solution is not readily apparent to a problem solver. Most word problems in textbooks are verbal translations of symbolic exercises that are transparent and easily solved without much struggle which make them routine translation problems. Textbooks are a significant factor influencing how teachers conduct their instruction, yet teachers may not have the resources to support students' work on complex problems if there are few problems within the textbook. The Standards for Mathematical Practice, Standards for mathematical content include learning outcomes related to problem solving. Students are expected to solve real-world problems, which are more complex than exercises. Recently adopted standards place a larger emphasis on problem solving in mathematics standards and instructors are not provided with resources indicating how to blend problem solving and mathematics into daily instruction so that students are prepared to solve realistic and complex problems drawing on current situations. Ideas for instruction stemming from research conducted in classrooms may support mathematics teachers to assist students to become effective problem solvers who solve complex problems as part of their day-to-day instruction. Students are expected to understand the approaches of others to solving complex problems. Thus, learning about alternate representations and procedures is critical to their success. Students learn several ways to solve problems over an academic year but it is not always clear whether a previously learned approach could be applied in a new situation or when one is more efficient than another. Developing productive problem-solving behaviors during classroom mathematics instruction includes promoting the idea that problems can be solved in multiple ways, often times using previously learned methods. Representation use is a critically important element of solving problems. Algorithms have been and continue to be a focus in many mathematics classrooms. They are important tools for solving mathematics problems and should be part of mathematics instruction but focusing mathematics instruction on learning algorithms does not orient students to determine the essential parts of a problem's situation or enhance their problem-solving performance. Instruction that allows students to manipulate tasks into more manageable or useful representations and employ a variety of representations and procedures facilitates students' development of mathematical proficiency. There is typically more than one way to solve a word problem. Implementing pictorial, tabular, or verbal representations to solve problems can often be more efficient while being just as effective as symbolic approaches. Much of the literature on students' representation use focuses on school level students. University students are capable of learning various ways to solve problems. They are able to recognize the limitations and benefits of mathematical representations and can develop strategic competence that makes them more efficient problem solvers. Unfortunately many findings from prior research that show improvements in students' representation use as a result of an intervention are not linked to learning required objectives from university standards. Hence, there is a need for research that characterizes mathematics instruction that supports students' use of a variety of representations within the context of these university mandated standards. Finally, prior investigations of problem-solving interventions have shown inconsistent effects on students' achievement. It has been widely found that university students who experienced one lesson each week that supported mathematics learning through problem-solving contexts had slightly better scores on an achievement than their peers experiencing traditional instruction. Conversely, incorporating problem solving into daily instruction resulted in negative or non-significant effects on some students' achievement. That is, the instructional intervention may influence the depth of students' understanding related to specific content areas. This study provides insight on students' knowledge of mathematics procedures and concepts related to different areas of mathematics by examining how an intervention group performs on a unit test and compares their outcomes to peers in a comparison group. In conclusion, research must continue to explore students' outcomes from teaching mathematics through problem-solving contexts. Results from studies examining problem solving, representation use, and achievement may provide insight into ways to support students' to become mathematically proficient problem solvers.

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5 PROBLEM-SOLVING PROCESS

The pathways individuals use to solve problems have been investigated extensively and various models of the problem-solving process have been proposed. Some mathematicians create a model of the problem-solving process that builds upon prior frameworks and identifies both appropriate and superficial pathways for solving problems. The pathways individuals use to solve problems have been investigated extensively and various models of the problem-solving process have been proposed. We create a model of the problem-solving process that builds upon prior frameworks and identifies both appropriate and superficial pathways for solving problems. The problem-solving process begins with an individual reading and understanding a problem's text. The text indicates the task and provides the reader with information about the problem. At times, the task is unclear from an initial reading of the problem so an individual rereads the problem. This requires being systematically active about his understanding so he can maintain engagement in the task. Understanding includes decoding the text into more manageable chunks in order to create a situation model. This situation model is the second stage in the problem-solving process. It is typically an internal representation encompassing mathematical, contextual, and other non-essential aspects of the problem, but some problem solvers create external representations of their models. This model is an intermediate model that links the problem's text and mathematical analysis phase of problem solving. As students discuss the problem, they form relational representations connecting internal and external representations. An example might clarify how situation models may be internal, external, or relational. Imagine a circumstance where an individual is given a text with an embedded problem. This person understands the text and creates a representation in his mind. In order for this individual to share situational model with another, the internal representation must be transformed into an external representation, which requires a relational representation. Regardless of whether an individual uses an internal, relational, and/or external representation to model the situation, effective problem solvers subsequently develop a more mathematically-focused model called a mathematical model and move to the third problem-solving stage. The mathematical model contains only mathematical aspects that can be acted on using mathematical analysis techniques. Some examples of mathematical models include graphs and pictures, symbolic expressions, tables, and verbal statements. Representation use during problem solving is crucially important if a student expects to find the correct solution. Effective problem solvers recognize that some representations are more appropriate or lead to the solution quicker than others, depending on the task. Furthermore, factors that might impact the mathematical model are more obvious to problem solvers who fully engage in the problem-solving process. Those who take the necessary time and energy to understand the text and develop a situation model are likely to solve the problem but that does not guarantee success. The present study examines students' representation use (i.e., mathematical modeling) within the context of the problem-solving process. Mathematical modeling is a critical step in the process because it leads to the mathematical analysis technique (i.e., procedures) used to answer the problem. The analysis procedures are dependent upon the mathematical model's representation. After implementing an analysis technique, the individual arrives at the fourth stage, derivations from analysis. This is not the final answer, but rather the outcome from carrying out a set of procedures on a mathematical representation. The derivation is just a number or another representation that has not had meaning ascribed to it by the problem solver. The result is important yet it needs to be interpreted within the problem's context. For example, word problems and real-life problems require units in order to make sense of the result. Effective problem solvers evaluate their result with the situation model, judge their alignment, and the outcome is the interpreted result. This evaluation requires a learner to self-monitor his mathematical thinking, being careful to consider whether the result aligns with the situation model and if not, to return to the appropriate problem-solving stage and reevaluate his work. Interpreting a problem's result is the fifth problem-solving stage. Problem solvers who externalize their situation models have something visible to verify their interpreted result whereas others have to revisit their working memory for the situation model. If the result aligns with the situation model then the problem solver communicates the answer. Reporting a solution is the sixth and final problem-solving stage. It occurs when a student effectively answers the questions such as by writing a summary statement or sharing the final solution with a peer. There are many steps to solving a mathematics problem and each stage is critically important to the learner's success. Successful problem solvers typically go through all six stages of the problem-solving process whereas unsuccessful problem solvers typically take at least one shortcut. Shortcuts are more likely to lead to inappropriate mathematical models, incorrect use of procedures, and reporting the wrong answer to the problem. Some of the common missteps are discussed here. At the first stage of the superficial problem-solving process, students read the text and create a mathematical model. This leap in the problem-solving process does not facilitate adequately understanding the text or determining the key aspects of the problem. At the third problem-solving stage, some learners employ mathematical representations that are inappropriate for a problem's context. For example, one may notice that some students often try using symbolic representations and algorithms to solve complex word problems. They are frequently unsuccessful and therefore we are justified to argue that if they had better facility with multiple representations then they might have shown better problem-solving performance. The role of mathematical representations is critically important for problem solver's success and it is a focus of this study. During mathematical analysis, learners often combine numbers inappropriately because they do not consider alternate representations or their situation model is inaccurate. Another common mistake is that problem solvers employ a representation, conduct procedures, and report the result as the problem's solution without interpreting it. For example, an individual might indicate 19 as a word problem's solution; however, the correct response requires meaningful units such as dollars, blocks, or people, etc. This expedited problem-solving process takes less time but it also leads to far more incorrect answers. A common error that can be made at any stage of the problem-solving process is not devoting the necessary cognitive energy to each stage of the process. One error made by many students is not taking time and cognitive energy to sufficiently understand a problem's text.

6 Understanding Text

Actively reading a problem supports individuals to make sense of it; however, the depth and quality of students' decoding and subsequent understanding of the text impacts their success. To solve a word problem, individuals must manage both the text and the mathematics encoded within the text. One's reading ability influences how likely an individual will solve a word problem and similarly, one's knowledge of mathematics influences how well an individual deciphers mathematics texts. Consequentially a subset of one's mathematical knowledge is one's ability to make sense of mathematics text. The depth and quality of understanding the text are two factors influencing how problem solvers approach a word problem. It is essential to sufficiently decode a problem's text into meaningful representations so that the task's elements are clear. Text difficulty can also impact how efficiently an individual solves a word problem. More difficult texts require more cognitive energy to decode, which may influence students' problem-solving behaviors. Prior studies have examined the influence of a problem's language on students' problem-solving behaviors and performance and provide evidence of students' troubles with word problems. It has been observed that most of the students prefer to use a direct translation approach, which includes (1) reading the problem, (2) executing a strategy, and (3) reporting the result. This approach does not foster success among participants with multistep and inconsistent language problems, but some will be able to solve straightforward consistent language problems in this fashion. This approach may suffice for simple word problems or translation tasks (e.g., symbolic expressions written as verbal statements) but it is insufficient for solving non routine or multi-step word problems. When tasks contain unfamiliar terminology or more words than typically seen in translation problems, students using this strategy may not adequately read and make sense of the problem. Reading and understanding a text influences which schemata are activated to solve the problem; hence this initial step in the problem-solving process is important results describe students' struggle with solving word problems, regardless of the problem's language and the representation and procedure employed. They often used ineffective approaches, which includes insufficiently reading and understanding a problem's text. Mathematics instruction should teach students to completely understand the problem before moving forward in the problem-solving process. The study provides an essential piece of the foundation for investigating students' problem-solving behaviors. There are other factors that contribute to problem-solving ability such as familiarity with mathematical terminology. Students with a sufficient understanding of conventional mathematics terminology are apt to solve problems because they understand the meaning of the words they read. Mathematics is a language that relies on symbols but it also includes graphs, charts, and texts to decode. Actually reading completely depends on being able to understand the structures of texts; to interpret authors' ideas; and to visualize, evaluate, and infer meanings to interpret authors' ideas. Recently evidence has begun to quantify the relationship between students' reading comprehension and problem-solving performance. The study has investigated the relationship between achievement on a test of word problems and reading comprehension. As a result the fairly strong positive correlations have suggested that students' success on each type of word problem (i.e., compare, change, combine, and focus) were associated with most aspects of reading comprehension as well as students' technical reading ability. Individuals with good reading comprehension skills are more likely to solve word problems than poor readers. Clearly, reading comprehension and mathematics knowledge are woven together, but further explorations with word problems that are not translation tasks are necessary. Such investigations verify whether students' reading comprehension impacts their ability to read and interpret word problems that do not follow word problems. If reading comprehension influences students' ability to solve simplified problems then it likely might impact students' ability to solve word problems. Investigations and analyses are necessary to confirm this hypothesis. Successfully solving a word problem depends on a problem solver's ability to initially read a problem's text and understand the task. If problem solvers effectively read and understand the text, which is the first problem-solving stage, then they are more likely to move to the second problem-solving stage, which involves creating an effective situation model.

7 Situation Modeling

The situation model is a representation of the text that characterizes the problem in a way that makes sense to the problem solver. It contains the mathematical and nonmathematical elements in a manageable representation that facilitate creating the mathematical model and clarifies the task. Adequate models support efficient and appropriate mathematical thinking leading towards a solution. These models are often internal representations, but sometimes learners formulate external representations such as drawing a picture of the situation or reconstructing the text using simpler words. Problem solvers need these representations to solve word problems because they clarify the task and necessary elements of the problem. Effective problem solvers often reread texts and make and revise their situation models before settling on one. Critical thinking skills supports individuals make decisions about the text and decode it into useful representations. More specifically, it is reasonable and reflective thinking that is focused on deciding what to believe or do, and is an important part of problem solving. Critical thinking requires students to reflect on how well they understand information and act on the information to create a model of the situation. Active online cognitive monitoring of task completion helps students make strategic adjustments toward reaching a desirable goal. Self-monitoring is also important while students create and refine their situation models. Students are apt to skip creating a situation model when they perceive it as unnecessary, especially for routine problems. Non routine word problems typically do not follow the typical language and structure associated with textbook word problems. They require students to read the text carefully and decode it into an adequate situation model. Instruction should encourage model creation regardless of the question's simplicity. Students may need assistance deciphering the text and determining what parts of the problem are important as well as what representations might be appropriate. The objective of instruction in this article is intended to benefit problem solvers by encouraging them to create adequate representations of the situation so that they could generate appropriate mathematical models. Representations are absolutely necessary for any mathematical activity to occur because mathematics typically uses sequences of symbolic characters that convey shared meanings among individuals. They provide means to link two or more configurations of an idea or concept. In the context of word problems, students create representations that (a) reproduce the action of a story problem; (b) strip away the context, attending only to numerical aspects of the problem; or (c) combine some of both approaches. Furthermore, as individuals or groups work on problems, they may make drawings, write notes, or construct tables or equations. These representations help them keep track of ideas and inferences they have made and also serve to organize their continuing work. They are one of the initial steps taken by problem solvers to proceed toward a solution. After carrying out the second problem-solving stage, effective problem solvers arrive at the third stage (i.e., mathematical modeling) and the subsequent fourth stage (i.e., derivations from analysis).

8 Mathematical Modeling, Analysis, and Derivations

This and next section combine two stages and an important process because the representation of the mathematical model influences what procedure is employed, which in turn impacts the derivation from analysis. A strategy includes the mathematical model (i.e., representation) and computational steps (i.e., procedures) hence it is necessary to characterize these stages and process together. Our study explicitly focuses on students' use of representations to solve word problems. Students' mathematical modeling` has been studied extensively for decades. Solving word problems requires thinking about possible mathematical models, selecting an appropriate representation for the situation, and determining the mathematical elements. Individuals with well-developed mathematical proficiency often consider multiple mathematical models before proceeding with one. Adequately describing problems in precise terms using mathematical models may take several iterations and practice but the payoff is worth the effort. Careful reexamination of previous mathematical models leads to more efficient problem solving on future tasks. Students' beliefs about doing mathematical activity influence their representation use. At times these beliefs and students' dispositions hinder their success or efficiency. Generating a mathematical model relies on several factors including an individual's comfort with different ways to represent the mathematical elements of a problem.

9 Representations

Representations include (a) experience-based scripts, (b) manipulative models, (c) pictures and diagrams, (d) graphs (e) verbal or spoken language, (f) written symbols, and (g) tables. Learners should be able to transform each type of representation into another one that is similar in representation but unique in other ways. Problem solvers can also translate a representation into another one (e.g., symbolic expression into verbal statement). Teachers can support problem solvers working on word problems by scaffolding students during the translation process. While many teachers use representations to help students learn mathematics, it is important that these instructor-generated representations are developed mentally appropriate. Overly complex representations beyond a learner's developmental grasp are likely to be confusing or inhibit future growth hence a teacher's scaffolding is critical to facilitating an individual's cognitive growth. Students who simply produce mathematical models and use representations without reflecting often misinterpret a task's goal or create insufficient mathematical models that fail to account for the key components of the problem. From an earlier, students can develop the misconception that solving mathematics problems should occur quickly and without having to reexamine the task or their models. Effective problem solvers select a representation and implement procedures based on their appropriateness for the context and efficiency. Allowing students to choose their own representation to solve a complex word problem provides a context for the entire class to examine ways of solving problems by using different mathematical models. Students are creative problem solvers, and when given rich tasks they are able to generate approaches that answer the question. Instruction should support students to learn nonsymbolic representations and require them to give a mathematically appropriate rationale for using a specific representation to mathematically model a situation. Many students think that representations are useful for a specific problem-type and rarely consider employing one representation to another problem-type. Furthermore, many students learn representation but do not necessarily know which one to implement unless cues or clues make it obvious. Students need to learn when to employ a representation, set of procedures, and gain strategic competence, to become effective and efficient problem solvers. Strategic competence is critical to a problem solver's success. Students need to know representations, suitable procedures for each representation, as well as when to use the representation. Specifically, it is the ability to formulate mathematical problems, represent them, and solve them. Problem solving helps to develop an individual's strategic competence because non routine or authentic tasks require a learner to consider multiple perspectives of the problem, select a viable representation, and perform the necessary steps to sufficiently carry out the procedures. Knowing multiple ways of solving a set of problems leads to individuals becoming more efficient and effective problem solvers. An individual who has developed strategic competence typically has a flexible approach for solving word problems. Students with well-developed knowledge of representations, procedures, and a conceptual understanding are likely to know how to solve problems using more than one representation, which includes employing the same mathematical model but using a different and perhaps more efficient set of procedures.

10 Multiple Representations

Students who know more than one way to solve a word problem are more likely to give the correct answer to a word problem. One needs to be caution that this knowledge of multiple approaches is useful only if individuals recognize the limitations of each method and have an understanding of when a representation and set of procedures are appropriate. To that end, one must develop strategic competence in order to efficiently solve problems. Many students are unsure how to proceed when presented with word problems. It has been noticed that some participants do not spend much time trying to understand the problem's text and typically use a symbolic representation. After quickly scanning the text, they immediately create a symbolic-oriented mathematical model without considering alternative representations. It has also been observed that some participants, they seem to employ direct translation approach fairly often. They lack adequate facility with multiple representations and struggle to produce more than one symbolic approach. More often participants allude to an inability to solve a problem because they cannot remember the necessary algorithm or formula. Participants indicate nonsymbolic as well as other symbolic approaches which might exist but are reticent to explore this possibility. Implementing a strategy requires knowledge about procedures and representations. Many students need opportunities to learn about alternatives to algorithms and formulas, the rationale for using various representations, and their associated limitations. An investigation such as the present study could help the teachers to foster effective problem-solving behaviors among students, including developing knowledge related to mathematical models, and improving students' problem-solving performance.

11 Interpreting and Reporting the Results

Interpreting results implies examining the outcome from an implemented strategy, thinking about what it means, and reflecting on the result's reasonableness given the problem's context, yet many ineffective problem solvers offer the outcome from a completed strategy as the final solution without interpreting it or reflecting on its appropriateness. Effective problem solvers are systematically active throughout the problem-solving process and typically decide whether the situation model and interpreted result align before completing the solution. Active cognitive self-monitoring helps learners make these judgments and successfully answer problems. Common errors such as reporting an incorrect response that is impossible given the context of the problem could be remedied by attention to this near final stage. Reporting the solution links both nonmathematical and typical mathematical language and facilitates the development of mathematically proficient students. It is the sixth and final problem-solving stage.

12 Summary of the Problem-solving Process

The present study is guided by a cognitive problem-solving framework. It characterizes the effective pathway for solving a problem as well as the superficial approach typically used by students. During the problem-solving process, individuals must also continue to be systematically active and pay attention to each stage of the process. The first phase of work toward solving a nonroutine word problem is reading and understanding the text. This stage means more than merely reading the text and moving to the next phase. It includes making sense of the problem's text and understanding the task. This leads to developing a situation model, which contains the mathematical and nonmathematical elements of the problem in more manageable problem representations such as pictures, diagrams, or verbal texts. After reflecting on the situation model and task, effective problem solvers construct and subsequently refine their mathematical models. This model captures the essential mathematical elements needed to solve the problem and facilitates working towards a solution. These models may appear in a variety of representations including symbolic, verbal, pictorial, graphical, and tabular and are the initial steps of implementing a strategy. Effective problem solvers construct and reflect on their mathematical models, which facilitate efficiently working toward the solution, and eventually a result. Moreover, they choose efficient representations that facilitate the development of situation and mathematical models during problem solving. Others rely on algorithms, heavy-handed approaches that may require significant cognitive effort, or know only one way to solve a problem. Interpreting the result requires reexamining the situation model and determining the solution's reasonableness given the problem's context.

If the solution seems appropriate, then the result should be reported in a clear statement that responds to the question. Research has shown that students have difficulty at many of these stages and often perform poorly on word problems. There is evidence that students can learn to develop productive problem-solving behaviors through instructional interventions, thus improving their problem-solving performance. Problem-solving instruction provides support to students so that they develop appropriate problem-solving behaviors, solve more exercises and word problems correctly, and consider a wide range of possible representations to use during the mathematical analysis phase. Tasks are a critical element of effective problem-solving instruction. Model Eliciting Activities (MEAs) have the potential to help students develop appropriate mathematical models and collaborate while solving open, complex, and realistic problems. The next section describes evidence from studies that implemented instruction aimed at improving students' problem-solving performance and representation use, which was the aim of the present study.

13 Problem-solving Instruction

Effectively engaging in the problem-solving process requires individuals to maintain their focus on a number of factors and work through each stage. Mathematics instruction that contains problem-solving elements can support students to engage in each stage of the process. Teachers often model the process or pathways to solve problems during instruction. Students generate valuable intuitive idealistic processes for solving problems. Effective teachers push their students to try alternative mathematical models while problem solving, and discuss successful and unsuccessful representations and procedures. Instruction should encourage students to understand the problem, create models, and consider multiple ways to solve problems. It should build upon students' prior knowledge and experiences and facilitate creating a network of mathematical topics, skills, and strategies. Problem-solving practice during mathematics instruction enhances students' mathematical understanding and in turn, well-developed mathematical understanding supports individuals to become more efficient and effective problem solvers. The next section characterizes how instruction can enhance students' problem-solving performance as well as their representation use during problem solving. Discussion of problem-solving instruction begins with investigations that supplemented everyday instruction with problem solving and transition to studies that blended problem solving and mathematics instruction to positively impact students' representation-use and problem-solving performance and behaviors.

14 Supplementing Daily Mathematics Instruction with Problem Solving

A number of instructional interventions supplement mathematics instruction with problem-solving elements in order to give students an opportunity to practice applying their mathematics knowledge to problems; hence word problems are called application problems at times. A typical lesson may begin with the teacher giving a problem and orchestrating a discussion focused on understanding the problem followed by students sharing possible ways to solve the problem. Next, students work independently or in small groups and finally, they share their representations, procedures, and solutions with the class. Participants give each other feedback, reflect on the problem-solving process, and share these reflections during the whole-class discussion. Skill activities and simple translation problems are completed individually whereas problem-solving activities are typically done in small-group interactions. Students may complete a pretest and posttest a few weeks later. Intermediate tests may be given to the intervention classrooms after a few weeks of problem-solving instruction. The tests may consist of problems and complex translation tasks. The teacher may score students' work on three dimensions using a three-point rubric: understanding of the problem, planning, and performance. Pretest scores may be used as a covariate in the analyses and each class is the unit of analysis. Some teachers say they like having the problem-solving process posted in the classroom and daily problem-solving activities. Some instructors comment that students are more frequently drawing a picture while problem solving, working backwards, creating a list of known information, and discussing problems with a peer during problem solving. This study provides evidence that problem-solving instruction enhances students' problem-solving behaviors and performance. It is the foundation for feasibility and other efficacy studies that examines students' outcomes from supplementing typical mathematics instruction with problem-solving components. Algorithmic teaching is an attempt to articulate traditional, textbook teaching prevalent in course. It deemphasizes the rationale behind procedures and representations unlike the other two instructional methods: meaning and problem-process. Meaning teaching emphasizes that students need to fully understand concepts and procedures and instruction attempts to facilitate students' cognitive connections between these two areas. Participants learn representations and procedures, but there are many more mathematically relevant student-to-teacher and student-to-student discussions in the meaning teaching classrooms than the algorithmic group. In the earlier study, students' performance in the meaning teaching group has been vastly higher than their peers experiencing algorithmic teaching. The third approach used is termed the problem-process approach. It incorporates a few minutes of daily problem-solving work into meaning teaching. The purpose of the later study is to determine whether there are any improvements in students' outcomes after supplementing meaning teaching. The three features of the problem-process instructional approach are (1) simple, content-related problems, (2) interactive discussions of solutions, and (3) a focus on the processes used to solve the problem. Students are frequently encouraged, to create models, craft explanations of their results, and provide multiple solution ways of problem solving. Students in both meaning-teaching and problem-process groups outperform peers in the algorithmic-process group on the achievement test. Average-achieving students in the problem-process and meaning-teaching group perform similarly. Low-achieving students in the problem-process group may have a few percent increase in their achievement over the algorithmic-practice teaching group whereas similar students in the meaning-teaching group make only a very little percent increase. These results inform the present study as well as others with ways to implement problem-solving instructional interventions as well as areas for improvement. These explorations may provide a substantial foundation for problem-solving research intending to use an instructional intervention, but several unanswered questions and gaps remain. One aspect that supports the present study is the lesson plan format implemented with participants. It includes use of small group discussions to foster mathematical thinking, scoring scheme, and analysis techniques for the study. One of the unanswered questions; however, is whether students' background characteristics or prior knowledge impacted their problem-solving performance. A second question stems from the idea of maintaining only a few minutes of problem-solving instruction. This surely limits the adequacy of influencing students' problem-solving behaviors and performance. The previous investigations offer evidence of participants' performance as measured by achievement tests and simple word problems, yet students' performance on open, complex, and realistic word problems is uncertain. Again, the research provides information about critical aspects for designing instructional interventions that focus on improving problem-solving performance and more insight about such interventions is provided in the discussion of further study. The researchers believe they might influence students' development of effective problem-solving behaviors by supplementing current instruction with problem-solving lessons. The purpose of the intervention is to make students (a) more aware of the different phases of the problem-solving process, (b) develop an ability to monitor and evaluate oneself during problem solving, and (c) master eight ways to solve problems. These problem-solving approaches are (1) draw a picture, (2) make a list or table, (3) distinguish necessary information from irrelevant material, (4) use real-word knowledge, (5) make a flowchart, (6) guess and check, (7) look for a pattern, and (8) simplify the numbers. Researchers designed two dozen lessons that were implemented over a one semester period. Lessons were guided by three pillars to construct a successful mathematics learning environment. The first pillar was that problem-solving instruction should use realistic, complex, and open problems. Problems related to students' real-life experiences so that solving these problems might feel meaningful. Complex problems facilitated engagement in the problem-solving process. Open problems permitted a variety of problem-solving approaches including multiple representations. For example, a simple translation task does not satisfy the first pillar. The second pillar was the use of a variety of instructional techniques including (a) short, whole-class introductions to the problem, (b) small-group collaborative problem solving, (c) individually completed independent work, and (d) concluding whole-class discussions to wrap up instruction and reflect on the concepts and skills learned that day. The final pillar was establishing a classroom culture with social and socio-mathematical norms for teaching and learning mathematics and problem solving. The authors describe the classroom culture as one that used (a) stimulating activities, (b) holding discussions with students about what counts as a good mathematics problem and response, as well as (c) appropriate mathematical procedures. Another feature of this culture was repositioning the role of teacher and students so that the teacher was not perceived as the holder of knowledge but rather a guide and mentor. The teacher was the driving force in developing these classroom norms for mathematics teaching and learning but the students were involved in making decisions as well. A before-during-after format study was employed. The teacher activated students' prior knowledge, engaged them in a problem-solving activity, and ended with an opportunity for reflection and synthesis. Researchers administered reliable versions of a pretest, posttest, and retention test composed of a dozen word problems that had similar problems across all three measures. The word problems were designed to be open, complex, and realistic. A few raters scored each test; responses were denoted as correct, incorrect, technical error, or no answer. A technical error meant that a student conducted the appropriate steps but made a mistake in procedures during problem solving. Because the aim of the study was students' engagement in the problem-solving process, technical error responses were considered to be correct. Students also completed an achievement test. Finally, a sample of a few lessons from intervention classrooms was selected to determine treatment fidelity. Students in both groups improved their problem-solving performance but the intervention group made greater gains. The intervention group also outperformed the control group on the achievement test. The addition of two dozen problem-solving lessons enhanced students' mathematics learning and problem-solving behaviors. Problem-solving investigators offer advice for researchers who plan to conduct problem-solving research in classrooms. Presumably, the results would have been better if we could have integrated the learning environment better within the regular mathematics lessons. That is, more consistent exposure to problem solving might have improved students' outcomes on the problem-solving measures. Students frequently encounter complex real-world problems, and solving exercises as well as problems will prepare them for these challenges. A concern with separating problem-solving instruction from everyday instruction as it had been done in the previous studies is the connotation that it presents to students: problem solving is distinct from mathematics. This issue informs the decision to integrate problem-solving instruction into typical daily mathematics instruction

15 Integrating Problem Solving into Daily Mathematics Instruction

The previous studies supplement typical mathematics instruction with components of problem-solving instruction. Results are fairly positive. Evidence did not indicate students' outcomes as a result of continuous mathematics instruction delivered through problem-solving contexts. To more fully integrate problem-solving instruction within typical mathematics teaching, the teacher aimed to determine the value of teaching mathematics in problem-solving contexts during a few weeks of investigation. In an earlier study, a few teachers immersed themselves in the classroom to better understand the teaching and learning process. They integrated components of problem solving into their daily mathematics instruction. Their study provided a foundation for becoming the instructor during an investigation as well as ideas for implementing effective mathematics and problem-solving instruction on daily basis. The teachers examined whether students might develop positive dispositions towards mathematical modeling as a result of experiencing mathematics instruction through problem-solving contexts. They encouraged students to think about problem solving as a multistage process and used word problems during instruction. The instructional intervention lasted approximately a few hours each day over a few continuous weeks at a university. One classroom was the intervention group while two similarly sized classrooms were the comparison group. The student-centered instruction typically incorporated discussion during its multiple phases. There was a central role of interactive and cooperative learning through small-group work and whole-class discussions. In phase one, students worked on a few problems in mixed-ability groups of a few students and then responded to some reflection questions. In phase two, the teacher and the students discussed the problem-solving process and the result from working on the problem. Students' misconceptions and ideas were explored so that the students could learn from each other. Students returned to their original groups and worked on approximately a few similar problems during phase three and then engaged in another whole-class discussion during phase four. During phase five, students solved some nonroutine word problems for homework that encouraged them to engage in the problem-solving process. A final whole-class discussion allowed students to share their reactions to the assignment and reflections completed the instructional process. With so many discussions, the teacher had to establish new social and socio-mathematical norms in the classroom if they were going to be effective. Norms are critically important for mathematical discourse to be productive. Relevant mathematical discourse among participants makes group activities during instruction worthwhile. It allows individuals to convey mathematical ideas and helps them to make sense of mathematical notions. Mathematically relevant communication helps build meaning and permanence for ideas and makes them public. Norms for discourse that encourage and perpetuate meaningful mathematics discussions must be established before it can have a lasting and meaningful impact on students' learning. Teachers influence students' use and type of communication based on the classroom norms and their behavior. Those who behave as the holder of knowledge or fail to allow students to justify solutions contribute to a learning environment where students are not responsible for their own learning. The teachers explain to students that their role and actions in the classroom might appear different than they had previously experienced, such as becoming a co-problem solver during lessons. They enacted norms that they believed might facilitate productive mathematical behaviors and dispositions. These socio-mathematical norms included determining what counts as (a) a good mathematical word problem, (b) a reasonable solution way to solve a problem, (c) an appropriate response, and (d) a satisfactory explanation. It is expected that students would show dramatic improvement in their problem-solving performance after learning in this environment. A pretest with a few word problems and a very few simpler translation tasks were administered, and participants completed a similarly constructed posttest. Participants in the intervention and comparison groups also completed a retention test a few weeks following the study. Students' responses were classified as realistic, nonrealistic, technical error, no answer, or other answer. A realistic response indicated a correct answer whereas no answer was an incorrect response, likely attributable to using the problem's information in an inappropriate manner. A technical error indicated that students answered the problem correctly except for a slight error. The other answer category was applied when a student's response could not be classified. To complement the data from the measures, one video recording of each classroom was made yet the tapes were not analyzed. Participants from the intervention group responded with realistic responses fewest number of times on the pretest but the most often on the posttest. Students in the intervention group improved but the higher performing students experienced the greatest benefits. Those who experienced these novel lessons continued to outperform their peers on the retention test a few weeks following the intervention. Students in the intervention group also developed positive dispositions towards nonroutine word problems whereas the control groups did not. This study provides evidence that students can learn to solve word problems, effectively engage in the problem-solving process, and develop positive dispositions towards word problems after a brief intervention. The research indicates that students quickly learn to adapt to a potentially novel learning environment and experience positive benefits from engaging in daily problem-solving instruction. This investigation also demonstrates that students can behave like mathematicians who regularly engage in problem solving. Feasibility studies provide a context for mathematics education researchers to better understand the teaching-learning process. Some teachers chose problems that encouraged students to think carefully and investigate multiple representations of a mathematical model and frequently reflect on the accuracy and appropriateness of the result from mathematical analysis. Students were encouraged to write their solutions on the board and the class was asked whether anyone wanted to question so-and-so's hypothesis. Solutions were hypotheses until they had been mathematically justified using mathematical language. The teachers, interactions with students were intended to promote that the teachers were a model of what it means to think like a mathematician. The teachers co-explored problems with students, engaged in mathematical discourse, and encouraged them to challenge their ideas and asked for justification. They used several instructional methods and often implemented a think-pair-share type of instruction. They posed a problem and took time to make certain each student understood the task and its text. Next, they observed students working independently and then listened to their peer-to-peer discourse about the problems. Finally, they initiated a whole-class discussion that clarified terms, symbols, and definitions for students, followed by collaborative problem exploration. They examined teaching episodes individually and then looked for patterns of change in students' outcomes. As a result of their instructions, students developed effective problem-solving behaviors, indicated more positive feelings about doing mathematics, and learned to work collaboratively to solve challenging problems. Their study provides clarity on a number of issues relating to the teaching and learning of mathematics. First, students are able to think and behave like mathematicians when given the opportunity. To attain this goal, the teacher must choose rich problems that require concentrated thinking and revising of ideas, and encourage collaboration and discourse that focuses on mathematics topics. Second, focusing on a few concepts provides a foundation for examining multiple ways to solve problems. When students master a few concepts during an academic year, they have opportunities to learn a wide variety of representations and procedures that likely will benefit their achievement and problem-solving performance in the long term. The teachers described students' thinking about the viability of multiple ways to solve problems as a result of their instructions. Their study delineates a rationale for making rich problems a focus of instruction as well as utilizing discourse to promote mathematics learning and effective problem-solving behaviors.

16 Model-eliciting Activities

Model-eliciting activities (MEAs) are related to the types of activities used in our study. These tasks are ill-structured, open-ended, complex, realistic tasks. They are typically small-group activities meant to support students' mathematical modeling for the present and future problems. An MEA differs from a traditional word problem in three ways: (1) the process and product are both important elements instead of the product only, (2) problem solvers can judge the adequacy of their mathematical model and solution by its relevancy and appropriateness for a problem's context whereas it may not be feasible with a word problem, and (3) the mathematical model is usually employed for other similarly structured problems whereas problem solvers might use the mathematical model for a word problems once. MEAs also precede formal mathematics instruction. Investigations have drawn on MEAs as a means to explore their effect on students' higher order thinking skills and metacognition. Furthermore, MEAs often connect students' knowledge of mathematics, other disciplines, and outside of school investigated students' outcomes as a result of implementing an MEA with one class of students. The activity's purpose was to support students to extend, explore, and refine ideas gained while solving previous modeling problems. Students were presented with a situation about a summer reading program for university students as well as a data set, asked to determine an appropriate solution, and finally explained and justified their response in a verbal statement. Problem solvers worked in groups of a few students over one week of class periods. Participants created pictorial models, which helped them develop symbolically-oriented mathematical models. They also showed an ability to quantify elements of a context in order to solve the problem. The problem's aspects included the length of a text and the text's readability level. Students solved the open, complex, and realistic problem and the lesson provided evidence that it is possible to weave problem solving, mathematics, and other subject areas. The study advocates that MEAs and other similar problem-solving activities should not be viewed as additional activities that add further load to an already crowded curriculum and overburdened teacher. They should be used to introduce, develop, consolidate, and enrich core concepts and processes. The present study heeded this message and participants experienced MEA-type activities as part of their daily mathematics instruction rather than as a supplement.

17 Summary of Problem-solving Instruction

Several studies have shown that supplementing mathematics instruction with problem-solving aspects enhanced students' problem-solving performance and assisted with developing productive problem-solving behaviors. Supplementing mathematics instruction looks slightly different across studies. The intervention group teachers devoted partial time of instruction to problem solving. A subsequent study indicated that dedicating partial instructional time was also sufficient for improving students' problem-solving performance. Results from this study also indicated that outcomes are not necessarily positive for all students. More specifically, average-achieving and above average-achieving participants from the problem-process group had lower achievement scores than their peers in the other groups. Finally, another group of students provided more correct responses to word problems after experiencing several problem-solving lessons over the course of an academic year than their peers in comparison classrooms. Furthermore, the intervention participants had better achievement scores than their peers, contrary to earlier findings. These studies provide evidence that problem-solving interventions generally produce positive problem-solving outcomes but achievement-related effects are more uncertain. There is also some evidence that integrating problem solving into daily mathematics instruction positively influences adolescents' problem-solving performance. Another group of teachers encouraged their students to discuss and critique each other's ideas about solving complex problems. Moreover, they incorporated problems into daily instruction so that students had rich problems that could be solved in multiple ways. Similarly, another group of teachers employed complex, open, and realistic problems as part of their mathematics instruction. Both investigations provide evidence that teaching mathematics through problem-solving contexts on a regular basis supports students' problem-solving performance. Several researchers have explored students' experiences with MEAs. MEAs provide experience working with open, complex, and realistic problems and support students' problem-solving development indicates that MEAs can be used as the central component of mathematics instruction, precede formal explicit instruction, and are not intended to supplement typical mathematics instruction. These rich activities when implemented in the context of discourse-rich, student-centered mathematics instruction lead to improving students' problem solving, engagement in mathematical modeling, and a number of other positive outcomes. Detailed analyses are necessary to convey to the research and teaching communities what changes occurred in students, possible explanations for the changes, and any differential effects of these changes. As indicated in the introduction, changes in students' representation use were examined in the present study. Detailed analyses are necessary to convey to the research and teaching communities what changes occurred in students, possible explanations for the changes, and any differential effects of these changes. Students' representation use is an indicator of whether students are engaging in effective problem-solving behaviors and thinking. None of the studies described thus far aimed to improve the number and types of representations used by students. The next section discusses the limited research conducted in university classrooms within the area of instruction promoting multiple ways to solve problems.

18 INSTRUCTION FOCUSING ON MULTIPLE REPRESENTATIONS

Students who are able to implement more than one representation to solve a word problem are more likely to solve it than peers who know only one way. Knowing multiple ways to solve a class of problems also indicates that a learner has developed strategic competence in an area. Two instructional programs were created around the empty number line representation, implemented them in matched classrooms, and examined student-related outcomes. One instructional program called the Realistic Program Design (RPD) was created so that students were encouraged to share their ideas and investigate multiple representations and procedures. Participants learned about the strengths and weaknesses of these representations in a discourse-rich, student-centered instructional environment. Word problems used during instruction drew on contexts relevant to students. The RPD program featured whole-class instruction and discussion about representations and procedures to solve word problems during approximately one third of the instructional period. In the other instructional program, Gradual Program Design (GPD), the teacher taught students how to use one representation and procedure at a time. There was less discussion in the GPD classroom and students spent more time completing exercises than problems. Word problems were a part of the GPD program but were treated as opportunities to practice applying a known representation and set of procedures. It was hypothesized that the GPD program might benefit the entire class' achievement more whereas the RPD program might improve low-ability students' achievement. The programs were implemented in several classrooms at different comparable universities. Classes were matched based on their prior achievement and randomly assigned to GPD and RPD conditions. A few mathematics test composed of several tasks were administered on different occasions several weeks apart. Students were asked to solve each problem on each test and show their work. Paired t-tests were used to determine whether there were any differences between students' outcomes on the tests. The interventions did not necessarily result in differential performance but there were differences in students' problem-solving behaviors. Students in the RPD program changed their use of procedures according to the characteristics of the problem whereas participants in the GPD group tended to use the taught representation and procedure, indicating that RPD participants developed greater strategic competence as a result of the instruction. There were almost no differences in procedural competence between the two groups of pupils. When significant differences were found, they were mostly in favor of RPD's pupils. At the end of the academic year, students in the GPD program still lagged far behind the RPD pupils in [strategic] flexibility and the RPD pupils attained and sustained a higher level of flexible problem solving than did the GPD pupils. This investigation provides evidence that instruction encouraging multiple ways to solve tasks leads to university students' development of effective problem-solving behaviors.

19 SUMMARY OF THE PROBLEM-SOLVING LITERATURE

Problem-solving instruction can enhance students' problem-solving performance and encourage developing and effective problem-solving behaviors. This instruction is best characterized by (a) problems that encourage critical thinking and reflection, (b) small-group and class-wide mathematical discussions, (c) opportunities for students to try different representations and talk about possible methods of analysis, (d) a teacher who behaves more like a facilitator and co-problem solver than someone who disseminates mathematical ideas and (e) students frequently justifying their ideas to one another rather than waiting for the teacher's confirmation. Many of these studies described establishing socio-mathematical norms in the classroom, which are crucially important to educators who expect their students to discuss mathematics and make sense of new ideas from this peer-to-peer discourse. Discussions about the tasks and possible representations involved in working towards a solution supported students' use of multiple representations, improved participants' mathematics achievement, and enhanced ability how students solve problems. These studies provide evidence that it is feasible to implement instruction that teaches mathematics content and problem solving while also supporting learners to develop strategic competence and become comfortable with using multiple representations to solve word problems. A necessary step forward for this type of research is conducting this type of instruction and drawing on the Standards. Students in classrooms where instruction encouraged multiple representation use were more efficient problem solvers, and at times more effective than their peers experiencing traditional instruction. The review of relevant literature informs the present study that aims to improve university students' problem-solving performance and representation use.

20 CONNECTIONS

This section will address the connections between prior literature and the present study, gaps and limitations of problem-solving research, and ways the present research managed these issues. Investigations into problem solving often draw on several aspects of the problem-solving process. An examination includes several steps of the problem-solving process and so this research greatly informs the research and teaching community about how students read and understand a problem's text. Reading and understanding text is important. A strong correlation between reading comprehension and problem-solving performance justifies including reading comprehension as a covariate in future problem-solving performance analyses. The classrooms should be selected so that differences between the intervention and comparison group are minimized. As a result of their work, several covariates should be included in the regression analyses. The focus of the present investigation is to implement an instructional intervention to enhance university students' problem-solving performance and representation use. The literature informs ways to maintain high quality research as well as areas for improvement. First, prior research suggests that approximately two dozens of lessons are sufficient to improve students' problem-solving performance and behaviors. The present study demonstrates ways to align mathematics standards and mathematics instruction with a problem-solving focus. There are positive outcomes after two dozens of lessons that were periodically delivered to students over a period of one semester. University students improved their problem-solving behaviors fairly quickly as well, even when problem-solving instruction was limited to a short time each day and separated from everyday instruction. As a result, there is some evidence that even brief and limited interventions support students' problem-solving performance. The resources indicating how to blend problem solving and mathematics into daily instruction so that students are prepared to solve realistic and complex problems draw on current situations. Ideas for instruction stemming from research conducted in classrooms may support mathematics teachers to assist students to become effective problem solvers who solve complex problems as part of their day-to-day instruction. Students are expected to understand the approaches of others to solving complex problems. Thus, learning about alternate representations and procedures is critical to their success. Students learn several ways to solve problems over an academic year but it is not always clear whether a previously learned approach could be applied in a new situation or when one is more efficient than another. Developing productive problem-solving behaviors during classroom mathematics instruction includes promoting the idea that problems can be solved by using multiple ways and often times using previously learned methods. Representation use is a critically important element of solving problems. Algorithms have been and continue to be a focus in many mathematics classrooms. They are important tools for solving mathematics problems and should be part of mathematics instruction, but focusing mathematics instruction on learning algorithms does not orient students to determine the essential parts of a problem's situation or enhance their problem-solving performance. Instruction that allows students to manipulate tasks into more manageable or useful representations and employ a variety of representations and procedures facilitates students' development of mathematical proficiency. There is typically more than one way to solve a word problem. Implementing pictorial, tabular, or verbal (i.e., non-symbolic) representations to solve problems can often be efficient and just as effective as symbolic approaches. University students are capable of learning various ways to solve problems. They are able to recognize the limitations and benefits of mathematical representations and can develop strategic competence that makes them more efficient problem solvers. Unfortunately many findings from prior research that show improvements in students' representation use as a result of an intervention are not linked to learning objectives from university standards. Hence, there is a need for research that characterizes mathematics instruction that supports students' use of a variety of representations within the context of university standards. Finally, prior investigations of problem-solving interventions have shown inconsistent effects on students' achievement. It has been found that university students who experienced two to four lessons each week that supported mathematics learning through problem-solving contexts had obtained slightly better scores on an achievement test than their peers experiencing traditional instruction. Conversely, incorporating problems solving into daily instruction resulted in negative or non-significant effects on some students' achievement. That is, the instructional intervention may influence the depth of students' understanding related to specific content areas. This study provides insight on students' knowledge of mathematics procedures and concepts related to mathematical analysis by examining the question how an intervention group performs tests and compares their outcomes to peers in a comparison group. In conclusion, research must continue to explore students' outcomes from teaching mathematics through problem-solving contexts. Results from studies on examining problem solving, representation use, and achievement may provide insight into ways to support students' to become mathematically proficient problem solvers.

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