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Improvements of two fatigue criteria based on material parameters

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ABSTRACT: Most of mechanical components in the engineering are frequently subjected to the fatigue damaging process because of the great number of stress cycles they have to undergo. This paper presents an elaboration of two of the most studied methods for the computation of fatigue life in multiaxial fatigue. We describe the reformulation of Sines and Crossland fatigue criteria, which have been adapted so that they could preserve the observed detrimental influence of a tensile mean stress and the observed beneficial effect of a compressive mean bending stress. The proposed reformulation of Crossland and Sines criteria is applied to a general sinusoidal in-phase or out-of-phase bending and torsion stress state, for which analytical formulae can be derived. From the theoretical results calculated according to the present propositions, the criterion proposed by Sines was found to be the most precise in preserving the detrimental influence of a tensile mean stress and the observed beneficial effect of a compressive mean bending stress. On the other hand, the criterion proposed by Crossland was found to be precise in the multiaxial fatigue limit prediction. This analysis shows that the proposed procedure is very efficient, suggesting that Sines and Crossland fatigue criteria remain valuable fatigue evaluation tools for the mechanical design industry.

KEYWORDS: high-cycle fatigue, mean stress effect; fatigue limit; proportional loading, non-proportional loading.

1 INTRODUCTION

A major part of mechanical components in the engineering is subjected to fluctuating loads, which can lead to sudden fatigue failure phenomenon. The durability analysis of these components and structures against fatigue is nowadays a main checking point of the design-engineering field [1], [2], [3].

Multiaxial fatigue criteria are generally associated to the estimate of the fatigue strength under complex loading. Hence, efficient and accurate methodologies for the account of the main factors influencing the fatigue strength of materials (type of loading, temperature, microstructural heterogeneities, residual stresses) under multiaxial stress states are required for use in engineering design application [4], [5], [6].

Many formulation of fatigue criteria have been proposed over years [2], [7]. Two of them that have been widely studied are those proposed by Sines [8] and Crossland [9]. These criteria are well-known criteria and are attractive for engineering design of high cycle fatigue components because easy-to-use; however these solutions are generally the weakest methods; unsafe when used for complex stress states. The main limitation in using the best solutions see [2], [7], [10], in situations of practical interest is that their application requires the definition of nominal parameters such as reference section, nominal stress, equivalent amplitude, etc. Lengthy and quite complicated calculations are required. Crossland and Sines criteria had over years been widely studied for a large number of loading cases: tension, bending, torsion, and combined tension/torsion, in phase or out of phase.

Sines [8] postulated that a mechanical component is in its fatigue limit condition when the following condition is assured:

$$\sqrt{J_{2a}} + \alpha_S \sigma_{H,max} \le \beta_S . \qquad (1)$$

where α_s, and β_s are material constants which can be calculated considering two fatigue limits. The errors between experimental data and the predictions of the Sine's criterion under non-proportional loading are bigger than for proportional loading; dependent on the material [10]. The Sines criterion provides non-conservative predictions for brittle materials [2].

Crossland [9] proposed a popular high-cycle fatigue criterion by considering that the influence of the hydrostatic stress must appear in the fatigue formula by its maximum value.

$$\sqrt{J_{2a}} + \alpha_C \sigma_{H,max} \le \beta_C \qquad (2)$$

This criterion is not sensitive to the detrimental effect of non-zero out-of-phase angles [2], [7]. Using the Papadopoulos minimum circumscribed hyper-ball approach [7] for fatigue damage evaluation, it appears that for biaxial tension with high mean stress, this criterion is too conservative.

The formulation of the Sines criterion is very close to the Crossland criterion, and only the mean stress effect incorporation differs. To preserve the observed detrimental influence of a tensile mean stress, parameters α_s, and α_c are required to be non-negative for a given material; this condition gives the range of applicability of these criteria [11]. A study of the formulation of these two criteria reported in the literature [7], [11] have shown that these two criteria does not give positive values for material parameters as required by the condition of validity of the criteria. Also, the fatigue life prediction widely underestimate the fatigue life [7], [2], [10].

The purpose of this contribution is to present a fatigue analysis procedure in the study of Crossland and Sines criteria, when adapted so as to preserve the observed detrimental influence of a tensile mean stress and the observed beneficial effect of a compressive mean bending stress. For this task the criteria are modified by virtue of a stress transformation that yields equivalent criteria when used to predict the fatigue strength of components subjected to combined tension/torsion, in phase or out of phase cyclic loading.

2 BACKGROUND

The Sines criterion [8] is one of the oldest and best-known criteria [2]. This criterion is written using the fatigue strength proposed in [12] as

$$E_S = \frac{\sqrt{J_{2a}} + \alpha_S \sigma_{H,mean}}{\beta_S} . \qquad (3)$$

Where α_s and βs are material parameters derived from two simple uniaxial tests: the fully repeated bending limit f_o and the fully reversed torsion limit t_{-1},

$$\beta_S = t_{-1} \text{ and } \alpha_S = 6\frac{t_{-1}}{f_0} - \sqrt{3} . \qquad (4)$$

To preserve the observed detrimental influence of a tensile mean stress the parameter α_s should be non-negative. Sines criterion range of applicability is given by the condition $\alpha_s > 0$, i.e.

$$\frac{t_{-1}}{f_0} > \frac{1}{2\sqrt{3}} . \qquad (5)$$

In Eq. (1), $\sqrt{J_{2a}}$ is the amplitude of the square root of the second invariant of the alternating deviator stress tensor and $\sigma_{H,mean}$ is the mean hydrostatic stress. If the fatigue limit in fully repeated bending f_0 is not provided, the Smith-Watson-Topper (SWT) parameter defined in [13] as $f_0=f_{-1}\, 2^{0.5}$, is used.

Instead of the mean hydrostatic stress, the Crossland criterion considers the combination of the amplitude of the second invariant of the stress tensor deviator and the maximum hydrostatic stress. Crossland criterion is mathematically expressed through the fatigue strength as:

$$E_C = \frac{\sqrt{J_{2a}} + \alpha_C \sigma_{H,max}}{\beta_C} \quad . \tag{6}$$

The material parameters in Crossland criterion are defined as:

$$\beta_C = t_{-1} \text{ and } \alpha_C = 3\left(\frac{t_{-1}}{f_{-1}} - \frac{1}{\sqrt{3}}\right) . \tag{7}$$

$\sigma_{H,\,max}$ in the expression of Crossland criterion is the maximum hydrostatic stress; and f_{-1}, is the fatigue limits in fully reversed bending. To preserve the observed beneficial effect of a compressive mean bending stress the parameter α_c must be positive. Crossland criterion is valid when $\alpha_c > 0$, i.e.:

$$\frac{t_{-1}}{f_{-1}} > \frac{1}{\sqrt{3}} \ . \tag{8}$$

The preference of Crossland to use the maximum hydrostatic stress instead of the mean hydrostatic stress as Sines has a tremendous effect on the prediction when compared to experimental fatigue tests from literature.

3 MATERIAL AND METHODS

The material parameters α_S and α_c in the criteria proposed Sines and Crossland have to be positive to ensure that the observed detrimental influence of a tensile mean stress and the observed beneficial effect of a compressive mean bending stress is preserved [11]. However as reported in Table 1, for some materials, these constants in Crossland criterion are negative. The principle for the formulation of the adapted criteria presented in this section, is to modify the Sines and Crossland criteria, by introducing a function of √J2a, $\sigma_{H,max}$ / $\sigma_{H,mean}$ and material independent parameter λ, such that the condition of validity will now be λ dependent.

We then calibrate the modified criteria so as to ensure that the adapted formulation renders positive values of the criteria parameters, α'_S and α'_c. In Appendix A and B, derivation of the constants appearing in the adapted criteria are presented.

3.1 ADAPTED SINES CRITERION

Based on the above considerations, the following expression with the same fatigue damage indicator as E_s, was proposed for adapted Sines criterion:

$$E'_S = \frac{\sqrt{J_{2a}}\left(1 + \lambda_s\left(\frac{\sigma_{H,mean}}{\sqrt{J_{2a}}} - \xi\right)\right) + \alpha'_S \sigma_{H,mean}}{\beta'_S} . \tag{9}$$

The parameters α'_S and β'_c can be also obtained from two uniaxial fatigue limits t_{-1} and f_0 (see appendix A). The Sines criterion in Eq. (9), is an extension of the classical Sines criterion Eq. (3).

$$\beta'_S = t_{-1}(1 - \lambda_s \xi) \ ; \tag{10}$$

$$\alpha'_S = 6(1 - \lambda_s \xi)\frac{t_{-1}}{f_0} - \frac{3}{\sqrt{3}}\left(1 + \lambda_s\left(\frac{\sqrt{3}}{3} - \xi\right)\right) \quad . \tag{11}$$

For simplicity, ξ in Eq. (11) is chosen such that ξ =√3/3. In other words, $\xi = \sigma_{H,\,mean} / \sqrt{J_{2a}}$ under fully repeated bending. Thus the adapted Sines criterion is now expressed as:

$$E'_S = \frac{\sqrt{J_{2a}}\left(1+\lambda_s\left(\frac{\sigma_{H,mean}}{\sqrt{J_{2a}}}-\frac{\sqrt{3}}{3}\right)\right)+\alpha'_S\sigma_{H,mean}}{\beta'_S}. \tag{12}$$

The following restrictions

$$\begin{cases}\alpha'_S > 0\\ \beta'_S > 0\end{cases}; \tag{13}$$

give the new range of applicability of the criterion, and now ensure that α'_S is positive for any material, as far as the real λ is chosen so that

$$\lambda_s < \left(\sqrt{3}-\frac{1}{2}\frac{f_0}{t_{-1}}\right). \tag{14}$$

3.2 ADAPTED CROSSLAND CRITERION

Just as in the precedent section, the expression of the extended Crossland criterion is in the form

$$E'_C = \frac{\sqrt{J_{2a}}\left(1+\lambda_c\left(\frac{\sigma_{H,max}}{\sqrt{J_{2a}}}-\xi\right)\right)+\alpha'_C\sigma_{H,max}}{\beta'_C}. \tag{15}$$

Constants α'_c and β'_c are calculated based on material constants under fully reversed torsion t_{-1} and fully reversed bending f_{-1} (see appendix B).

$$\beta'_C = t_{-1}(1-\lambda_c\xi); \tag{16}$$

$$\alpha'_C = 3(1-\lambda\xi)\frac{t_{-1}}{f_{-1}}-\frac{3}{\sqrt{3}}\left(1+\lambda_c\left(\frac{\sqrt{3}}{3}-\xi\right)\right). \tag{17}$$

The restrictions on the values of α'_c and β'_c given in Eq. (16) provides the new range of validity of the adapted criterion, and ensures that α'_c is always positive.

$$\begin{cases}\alpha'_C > 0\\ \beta'_C > 0\end{cases} \tag{18}$$

For simplicity, the value of the real parameter ξ is taken to be √3/3 and the Crossland criterion now preserves the experimental observation that requires a positive value for α'_c. The real lambda have to satisfy

$$\lambda_c < \left(\sqrt{3}-\frac{f_{-1}}{t_{-1}}\right). \tag{19}$$

Thus the adapted Crossland criterion is simply expressed as:

$$E'_C = \frac{\sqrt{J_{2a}}\left(1+\lambda_c\left(\frac{\sigma_{H,max}}{\sqrt{J_{2a}}}-\frac{\sqrt{3}}{3}\right)\right)+\alpha'_C\sigma_{H,max}}{\beta'_C}. \tag{20}$$

4 RESULTS AND DISCUSSION

In other to show the accuracy of the adapted criteria to preserve the observed beneficial effect of a compressive mean bending stress or the observed detrimental influence of a tensile mean stress in estimating high cycle fatigue strength under multiaxial fatigue loading, results from a systematic bibliographical investigation on un-notched samples reported in [14] are used.

Table 1 summarizes the fatigue properties of the materials, that is the values α_C, α'_C, α_S and α'_S obtained from the original and adapted criteria. The value of parameter lambda in the computation was taken to be $\lambda_s = \lambda_c = -2$.

One can clearly see from Table 1 the negative values of the parameter α_C in Crossland criterion, obtained for different materials. These negative values are now made positive, α'_C, using the adapted Crossland criterion. The criterion proposed by Sines criterion is most precise in preserving the detrimental influence of a tensile mean stress and the observed beneficial effect of a compressive mean bending stress since the computed values of α_S, α'_S are positive.

Further, the accuracy of the new adapted fatigue criteria to coincide with the original formulations when predicting fatigue failure is determined by comparing the proximity of the predicted total damage *E*, to unity, Eq. (21). The predicted fatigue damage indicator *I* measures the relative difference between the estimation of the criterion and the experimental data. A negative value of the damage indicator *I* means that, the criterion predicts a greater fatigue limit than experimental one; resulting in a non-conservative prediction.

$$I = \frac{E-1}{1}\times 100\% \cdot \tag{21}$$

Conversely, a positive value of *I* corresponds to a conservative prediction. If the error index *I* is close to zero, it means that the agreement is good between prediction and experimental results. The error index *I* is expressed as:

Table 1. Material parameters of un-notched materials [14], where α_C is the Crossland material parameter, α'_C the adapted Crossland material parameter, α_s the Sines material parameter, α'_s the adapted Sines material parameter

Material	t_{-1}	f_{-1}	t_{-1}/f_{-1}	α_C	α'_C	α_s	α'_s
0.1% C steel (normalised)	151.3	268.6	0.56	-0.04	1.91	53.66	117.62
0.4% C steel (spheroidized)	155.9	274.8	0.57	-0.03	1.94	54.70	119.85
NiCrMoVa steel	342.7	660.7	0.52	-0.18	1.62	78.26	170.63
NiCr steel (Solid samples)	369.7	666.7	0.55	-0.07	1.85	84.18	183.37
NiCr steel (Hollow samples)	339.6	653.2	0.52	-0.17	1.63	77.99	170.05
0.34% C steel	218	378	0.58	-0.00	2.00	65.54	143.23
Mild Steel	137.3	235.4	0.58	0.02	2.04	51.86	113.96
St35	130	230	0.57	-0.04	1.92	49.70	109.09
XC18	186	332	0.56	-0.05	1.89	59.52	130.24
High strength steel	364	630	0.58	0.00	2.00	85.28	185.75

Table 2 represent relevant experimental results, available in the literature, concerning synchronous in-phase or out-of-phase sinusoidal loading experiments on different materials coming from [6]. The out-of-phase bending and torsion, as is known, is the starting point for the theoretical study of many researchers [4]. Therefore this loading case is considered in the following for the prediction of the fatigue limit.

The following non-proportional to proportional stress transformation that can improve stress invariants based criteria accurately for fatigue evaluation under out-of-phase multiaxial loading is used to compute the error index in Sines and Crossland criteria. Under bending-torsion sinusoidal signals, the time histories of the axial stress, $\sigma_{xx}(t)$, and of the shear stress, $\tau_{xy}(t)$, can be expressed as follows:

$$\begin{cases}\sigma_{xx}(t)=\sigma_{xx,m}+\sigma_{xx,a}\sin(\omega t);\\ \tau_{xy}(t)=\tau_{xy,m}+\tau_{xy,a}\sin(\omega t-\beta).\end{cases} \tag{22}$$

Where *m* identifies the mean value of the signals, *a* the amplitudes and, finally, β is the phase shift between the applied stress components, n is a material dependent parameter.
We proposed to take into account the effect of the non-proportionality of the applied load in the computation of Sines and Crossland criteria [3] ,[7], [10], by proposing the following equivalent fatigue proportional stress state of Eq. (22), as:

$$\begin{cases}\sigma'_{xx}(t)=\sigma_{x,m}+\sigma_{x,a}\,sin(\omega t);\\ \tau'_{xy}(t)=\tau_{xy,m}+\tau_{xy,a}\left(\left|cos\,\beta'+sin\,\beta'\right|\right)^{n}sin(\omega t);\\ \beta'=\left(\delta_{0\beta}-1+\beta\right);\\ n=f_{-1}^{\;-0.60}.\end{cases} \tag{23}$$

In the above expression, $\delta_{0\beta}$ is the kronecker delta defined for an arbitrary phase shift angle β as

$$\delta_{0\beta}=\begin{cases}1 \;\; if\;\;\beta=0\\ 0 \;\; if\;\;\beta\neq 0\end{cases}. \tag{24}$$

For the results of predictions grouped together in Table 2, the error index I_c *(%)*, refers to the results of Crossland criterion, I'_c *(%)* to the adapted Crossland criterion; I_s *(%)* to Sines criterion, I'_s *(%)* to the adapted Sines criterion.

Table 2. Fatigue strength of hard steel (f_{-1}=313.19 MPa, t_{-1}=196.2 MPa, R_m= 680 MPa)

$\sigma_{xx,a}$	$\sigma_{xx,m}$	$\sigma_{xy,a}$	$\sigma_{xx,m}$	β(°)	I_c (%)	I'_c (%)	I_s (%)	I'_s (%)
138.1	0	167.1	0	0	-2.3	-2.3	-5.6	-5.6
140.4	0	169.9	0	30	0.1	0.1	-3.3	-3.3
145.7	0	176.3	0	60	3.9	3.9	0.4	0.4
150.2	0	181.7	0	90	6.3	6.3	2.7	2.7
245.3	0	122.6	0	0	1.4	1.4	-4.5	-4.5
249.7	0	124.8	0	30	3.7	3.7	-2.4	-2.4
252.4	0	126.2	0	60	4.8	4.8	-1.3	-1.3
258.0	0	129.0	0	90	6.7	6.7	0.5	0.5
299.1	0	62.8	0	0	0.9	0.9	-6.3	-6.3
304.5	0	63.9	0	90	2.7	2.7	-4.7	-4.7

The predicted fatigue damage indicators using the original Crossland criterion I_c are equivalent to the values obtained using the proposed adapted Crossland criterion, I'_c. Similarly, the same agreement between I_s and I'_s values is observed using the Sines criterion. This results indicate that our proposition of the adapted criteria are successful in accounting the mean stress effect while preserving the good predictions for in phase proportional loadings. The non - proportional to proportional stress transformation was useful in accounting the non-zero-out-of-phase effects by the studied criteria.

From the results reported in Table 2, we calculated the average error index I^{avr} *(%)*. For the criterion proposed by Crossland, we obtained $I_c^{\;avr}$ *= 2.82 % (%)*; whereas for Sines criterion we had $I_s^{\;avr}$ *= -2.45 %***.** The positive value of the error index obtained from Crossland criterion ($I_c^{\;avr}$ *= 2.82 %)*, means that the criterion yields conservative results. Thus Crossland criterion with the non-proportional to proportional computation approach is the most successful in the fatigue limit prediction as widely reported in the literature for results obtained with the minimum circumscribed ellipse approach [6] and minimum circumscribed hyper-ball approach [7].

5 CONCLUSIONS

The proposed and tested method of adapted criteria seems to be easy in application. Its implementation give positive values for material parameters as required by the condition of validity of the criteria. The use of the adapted criteria to predict fatigue limits shows that the adapted criteria is an extension of the studied classical fatigue criteria as expected.

With the use of the non-proportional to proportional stress transformation, results reported in Table 2 shows that a stress invariant-based multiaxial fatigue criterion, such as the Sines criterion can be applied with improved accuracy for fatigue evaluation under out-of-phase multiaxial loading. The approach allow the Sines criterion to behave correctly under out-of-phase loading, by the reduction of the very large scatter of results due to the too strong phase shift effect, which is further increased by the improper mean stress effect [2], [7]. The method proposed in the present paper must be generalized for asynchronous fatigue loads with random stress waveforms. The other problem is the study of materials under fluctuating thermal loads, by using the proposed adapted criteria. These topics are in-progress.

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APPENDIX. A DETERMINATION OF THE CONSTANTS OF SINES CRITERION

The new constants of the Sines criterion are determined from the limit of fatigue in fully reversed torsion t_{-1} and repeated bending f_0.

For fully reversed torsion, one has

$$\sqrt{J_{2a}} = t_{-1} \; and \; \sigma_{H,m} = 0 \, . \tag{A.1}$$

From the application of the adapted criterion, given in Eq. (7), with a maximum allowable damage of unity one finds that

$$\beta'_S = t_{-1}(1-\lambda\xi). \tag{A.2}$$

For fully repeated bending test,

$$\sqrt{J_{2a}} = \frac{f_0}{2\sqrt{3}} \; and \; \sigma_{H,m} = \frac{f_0}{6} \, . \tag{A.3}$$

And the expression of parameter α'_S obtained from Eq. (7), assuming a maximum allowable damage of unity is given as

$$\alpha'_S = 6(1-\lambda\xi)\frac{t_{-1}}{f_0} - \frac{3}{\sqrt{3}}\left(1+\lambda\left(\frac{\sqrt{3}}{3}-\xi\right)\right). \tag{A.4}$$

APPENDIX. B DETERMINATION OF THE CONSTANTS OF CROSSLAND CRITERION

The identification of the parameters α'_c and β'_c of the adapted Crossland criterion, we needs the knowledge of two uniaxial fatigue limits, t_{-1} and f_{-1}.

From fully reversed torsion, one has

$$\sqrt{J_{2a}} = t_{-1} \; and \; \sigma_{H,\max} = 0 \, . \tag{B.1}$$

Application of the criterion, as expressed in Eq. (13) provides the material parameter β'_c as

$$\beta'_C = t_{-1}(1-\lambda\xi). \tag{B.2}$$

From fully reversed bending test,

$$\sqrt{J_{2a}} = \frac{f_{-1}}{\sqrt{3}} \; and \; \sigma_{H,\max} = \frac{f_{-1}}{3} \, . \tag{B.3}$$

One finds the expression of parameter α'_c using Eq. (13) as:

$$\alpha'_C = 3(1-\lambda\xi)\frac{t_{-1}}{f_{-1}} - \frac{3}{\sqrt{3}}\left(1+\lambda\left(\frac{\sqrt{3}}{3}-\xi\right)\right). \tag{B.4}$$