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Convexity of the Set of k-Admissible Functions on a Compact Kähler Manifold

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ABSTRACT: We prove in this article using some convex analysis results of A. S. Lewis, the log-concavity of spectral elementary symmetric functions on the space of Hermitian matrices, and the convexity of the set of k-admissible functions on compact Kähler manifolds.

KEYWORDS: Spectral functions, symmetric functions, log-concavity, convexity, admissible functions, hessian equations, Kähler manifolds.

1 INTRODUCTION AND STATEMENT OF RESULTS

All manifolds considered in this article are **connected**.

Let $(M, J, \mathrm{g}, \omega)$ be a compact connected Kähler manifold of complex dimension $m \geq 1$. Fix an integer $1 \leq \mathrm{k} \leq \mathrm{m}$. Let $\varphi : \mathrm{M} \to \mathbb{R}$ be a smooth function and let us consider the (1,1)-form $\tilde{\omega} = \omega + \mathrm{i}\,\partial\,\overline{\partial}\varphi$ and the associated 2-tensor $\tilde{g}$ defined by $\tilde{g}(X,Y) = \tilde{\omega}(X, JY)$. Consider the sesquilinear forms h and $\tilde{h}$ on $T^{1,0}$ defined by $h(U,V) = \mathrm{g}(\mathrm{U}, \overline{\mathrm{V}})$ and $\tilde{h}(\mathrm{U},\mathrm{V}) = \tilde{g}(\mathrm{U}, \overline{\mathrm{V}})$. We denote by $\lambda(g^{-1}\tilde{g})$ the eigenvalues of $\tilde{h}$ with respect to the hermitian form h. By definition, these are the eigenvalues of the unique endomorphism A of $T^{1,0}$ satisfying: $\tilde{h}(\mathrm{U},\mathrm{V}) = \mathrm{h}(\mathrm{U}, \mathrm{AV}) \quad \forall \mathrm{U},\mathrm{V} \in T^{1,0}$.

Calculations infer that the endomorphism A writes:

$$\begin{aligned} A :\ & T^{1,0} \to T^{1,0} \\ & U^i\partial_i \to A_i^j U^i \partial_j = g^{j\bar{l}}\tilde{g}_{i\bar{l}} U^i \partial_j \end{aligned}$$

A is a self-adjoint/hermitian endomorphism of the hermitian space $(T^{1,0}, h)$, therefore $\lambda(g^{-1}\tilde{g}) \in \mathbb{R}^m$.

Let us consider the following cone: $\Gamma_k = \{\lambda \in \mathbb{R}^m, \forall 1 \leq j \leq k \ \ \sigma_j(\lambda) > 0\}$, where σ_j denotes the j-th elementary symmetric function.

Definition. φ is said to be k-admissible if and only if $\lambda(g^{-1}\tilde{g}) \in \Gamma_k$.

In a note in the Comptes Rendus de l'Académie des Sciences de Paris published online in December 2009 [1], we solve the equations $\tilde{\omega}^k \wedge \omega^{m-k} = \frac{e^f}{\binom{m}{k}}\, \omega^m \ (E_k)$, when the holomorphic bisectional curvature of M is nonnegative. In this proof performed by the continuity method, two results following from convex analysis techniques were needed, namely the Corollaries 1.5 and 1.6.

Let us now introduce some convex analysis notations. Let $H_m(\mathbb{C})$ be the space of complex Hermitian matrices of order m. We recall that for any two matrices B and C of $H_m(\mathbb{C})$, $\lambda \in \mathbb{C}$ is called a B-eigenvalue of C if there exists $x \neq 0$ in $\mathbb{C}^m$ such that $Cx = \lambda Bx$, x is then called a B-eigenvector of C. Let $B \in H_m(\mathbb{C})$ be **a fixed** positive definite matrix. Let us recall the following basic result :

Lemma 1.2. Let $C \in H_m(\mathbb{C})$, then:

1. The spectrum of $B^{-1}C$ (i.e. the B-spectrum of C) is entirely real.
2. The greatest eigenvalue of $B^{-1}C$ (i.e. the greatest B-eigenvalue of C) equals $\sup_{u\neq 0} \frac{<Cu,u>}{<Bu,u>}$, where $<.,.>$ denotes the standard Hermitian product of $\mathbb{C}^m$.

3. $B^{-1}C$ is diagonalizable.

Since the spectrum of $B^{-1}C$ is the spectrum of the Hermitian matrix $B^{-\frac{1}{2}}CB^{-\frac{1}{2}}$, the proof is an easy adaptation of the standard one for symmetric matrices.

For a given Hermitian matrix C, we denote by $\lambda_B(\mathrm{C})$ the eigenvalues of C with respect to B. In this article, we prove the following four results using the Theorem 2.3 and the Corollary 2.4 of Lewis [2] (see Theorem 2.1 of this article):

Theorem 1.3. For each $k \in \{1, \dots, \mathrm{m}\}$, the function:

$$F_k^B: H_m(\mathbb{C}) \to \mathbb{R} \cup \{+\infty\}, C \to F_k^B(C) = \begin{cases} -\ln \sigma_k\big(\lambda_B(C)\big) \; if \; C \in {\lambda_B}^{-1}(\Gamma_k) \\ \qquad +\infty \text{ otherwise,} \end{cases}$$

where ${\lambda_B}^{-1}(\Gamma_k) = \{C \in H_m(\mathbb{C}),\ \lambda_B(\mathrm{C}) \in \Gamma_k\}$, is convex.

Theorem 1.4. If Γ is a (non empty) symmetric convex closed set of $\mathbb{R}^m$, then ${\lambda_B}^{-1}(\Gamma) := \{C \in H_m(\mathbb{C}),\ \lambda_B(\mathrm{C}) \in \Gamma\}$,

is a convex closed set of $H_m(\mathbb{C})$. In particular, ${\lambda_B}^{-1}(\overline{\Gamma_k})$ is a convex closed set of $H_m(\mathbb{C})$.

By the Theorem 1.3, and since ${\lambda_B}^{-1}(\Gamma_k)$ is convex (Theorem 1.4), we deduce that:

Corollary 1.5. The function:
$-F_k^B: {\lambda_B}^{-1}(\Gamma_k) \to \mathbb{R}, C \to F_k^B(C) = \ln \sigma_k\big(\lambda_B(C)\big)$ is concave.

The method used here to prove the Corollary 1.5, gives for $B = I$ a different approach from the proof of [3] and the elementary proof of [4, p. 51] and [5].

As an immediate consequence of Theorem 1.4, we get the following important result, that allows to notably simplify the proof of uniqueness of the solution of the equation (E_k) in comparison with [5]:

Corollary 1.6. For a compact connected Kähler manifold $(M, J, \mathrm{g}, \omega)$, the set of k-admissible functions:
$\mathrm{A}_k = \{\varphi \in C^2(M, \mathbb{R}),\ \lambda_\omega\big(\omega + \mathrm{i}\,\partial\,\bar{\partial}\varphi\big) \in \Gamma_k\}$ is convex.

2 SOME CONVEX ANALYSIS

The space $H_m(\mathbb{C})$ has a structure of **Euclidean space** thanks to the following scalar product $\ll A, B \gg = \mathrm{tr}({}^{\mathrm{t}}\bar{\mathrm{A}}\,\mathrm{B}) = \mathrm{tr}(\mathrm{AB})$, called the **Schur product**. Let us denote by $\Gamma_0(\mathbb{R}^m)$ the set of functions $u: \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ that are convex, lower semicontinuous on $\mathbb{R}^m$, and finite in at least one point. Given $u \in \Gamma_0(\mathbb{R}^m)$ symmetric and $B \in H_m(\mathbb{C})$ **positive definite**, we define:

$$V_u^B: H_m(\mathbb{C}) \to \mathbb{R} \cup \{+\infty\}, by\ C \to V_u^B(C) := u(\lambda_{B,1}(C), \dots, \lambda_{B,m}(C))$$

where $\lambda_{B,1}(C) \geq \lambda_{B,2}(C) \geq \dots \geq \lambda_{B,m}(C)$ denote the **B-eigenvalues** of C repeated with their multiplicity. Such functions V_u^B are called **functions of B-eigenvalues** or **B-spectral functions**. Our first aim is to determine the conjugation for such a function V_u^B using the conjugate function of u. Let us remind that the conjugation or the Legendre–Fenchel transform of u is the function $u^*: \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ defined by:

$$\forall s \in \mathbb{R}^m,\ u^*(s) = \sup_{x \in \mathbb{R}^m}\{\langle s, x\rangle - u(x)\}$$

where $\langle .,. \rangle$ denotes the standard scalar product on $\mathbb{R}^m$.

Theorem 2.1 (A. S. Lewis [2], Conjugation of spectral functions). Let $u \in \Gamma_0(\mathbb{R}^m)$ be symmetric, then:

1. The conjugate u^* ($\in \Gamma_0(\mathbb{R}^m)$) is also symmetric.

2. The functions of eigenvalues V_u^I and $V_{u^*}^I$ (defined as above) belong to $\Gamma_0(H_m(\mathbb{C}))$ with $V_{u^*}^I = (V_u^I)^*$, so that in particular the function of eigenvalues V_u^I is convex and lower semicontinuous.

Proof. See the Theorem 2.3 and the Corollary 2.4 of Lewis [2]. ■

A similar theorem is proved in the case of symmetric matrices in [6] and [7] (you can see also [8] and [9] for some details).

Corollary 2.2 (Conjugation of B-spectral functions). Let $u \in \Gamma_0(\mathbb{R}^m)$ be symmetric, then:
1. The conjugate u^* ($\in \Gamma_0(\mathbb{R}^m)$) is also symmetric.
2. The functions of B-eigenvalues V_u^B and $V_{u^*}^B$ (defined as above) belong to $\Gamma_0(H_m(\mathbb{C}))$ with $V_{u^*}^{B^{-1}} = (V_u^B)^*$, so that in particular the function of B-eigenvalues V_u^B is convex and lower semicontinuous.

3 PROOF OF THEOREMS 1.3 AND 1.4.

3.1 PROOF OF THEOREM 1.3.

The proof of Theorem 1.3 is a direct application of the Corollary 2.2 to the function:

$$\mathrm{u} : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}, x = (x_1, \dots, x_m) \to \mathrm{u}(\mathrm{x}) = \begin{cases} -\ln \sigma_k(x_1, \dots, x_m) \ if \ x \in \Gamma_k \\ +\infty \ otherwise \end{cases} \quad (3.1)$$

Our function u is symmetric and belongs to $\Gamma_0(\mathbb{R}^m)$, indeed:

(i) It is clearly symmetric. It is finite in a least one point of $\mathbb{R}^m$ because Γ_k is non empty. And it is convex, because the function $(\sigma_k)^{\frac{1}{k}} : \Gamma_k \to \mathbb{R}$ is concave [3, p.269].

(ii) It is lower semicontinuous. Indeed, let $c \in \mathbb{R}$, and consider the set:

$$\begin{aligned} \{x \in \mathbb{R}^m / +\infty \geq u(x) > c\} &= \{x \in \Gamma_k \,/\, u(x) > c\} \cup \{x \notin \Gamma_k / u(x) > c\} \\ &= \{x \in \Gamma_k / -ln\, \sigma_k(x) > c\} \cup (\mathbb{R}^m \setminus \Gamma_k) \quad (3.2) \end{aligned}$$

By continuity, $\{x \in \Gamma_k \,/ -ln\, \sigma_k(x) > c\}$ is an open set of Γ_k, it is then an open of $\mathbb{R}^m$ since Γ_k is an open of $\mathbb{R}^m$. Furthermore, the cone Γ_k is also a closed set of $\mathbb{R}^m$ (as a connected component), consequently $\mathbb{R}^m \setminus \Gamma_k$ is an open set of $\mathbb{R}^m$. Therefore, $\{x \in \mathbb{R}^m / +\infty \geq u(x) > c\}$ is an open set of $\mathbb{R}^m$ too. This is valid for all $c \in \mathbb{R}$, so that u is lower semicontinuous.

Therefore, we deduce by the Corollary 2.2 that the B-spectral function V_u^B=F_k^B is <u>convex</u>, which proves the theorem.

Let us remark that the same technique allows to prove for example that the functions

$V(C)$:= "the greatest B-eigenvalue" of C and
$V_s(C)$:= "the sum of the s greatest B-eigenvalues" of C
with $s \in \{1, \dots, m\}$, (3.3)

are convex on $H_m(\mathbb{C})$.

3.2 PROOF OF THEOREM 1.4.

The proof of Theorem 1.4 goes by considering the indicatrix function $f_0 := I_\Gamma$ of the set Γ, namely:

$$f_0 := I_\Gamma : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}, x = (x_1, \dots, x_m) \to I_\Gamma(x) = \begin{cases} 0 \ if \ x \in \Gamma \\ +\infty \ otherwise \end{cases} \quad (3.4)$$

From the assumptions made on Γ, f_0 lies in $\Gamma_0(\mathbb{R}^m)$ and is symmetric, indeed:

(i) This function is clearly finite in at least one point since Γ is non empty.

(ii) The inequality $I_\Gamma(tx + (1-t)y) \leq t\, I_\Gamma(x) + (1-t)\, I_\Gamma(y)$ is valid for all $x, y \in \mathbb{R}^m$ and all $t \in [0,1]$. Indeed, if $x, y \in \Gamma$ then $tx + (1-t)y \in \Gamma$ by convexity of $\Gamma$and the two sides of the convexity inequality equal 0 in this case. Furthermore, if x or y does not belong to Γ then the right side of the inequality equals $+\infty$ and the inequality is then satisfied in this case too, which proves that I_Γ is convex.

(iii) I_Γ is lower semicontinuous. Indeed, let $a \in \mathbb{R}^m$: If $a \geq 0$ then $\{x \in \mathbb{R}^m / +\infty \geq I_\Gamma(x) > a\} = \mathbb{R}^m \setminus \Gamma$ is an open set since Γ is closed. Besides, if a < 0, $\{x \in \mathbb{R}^m / +\infty \geq I_\Gamma(x) > a\} = \mathbb{R}^m$ is an open set too.

So Corollary 2.2 implies that the function of B-eigenvalues $V_{I_\Gamma}^B$ lies in $\Gamma_0(\mathbb{R}^m)$; in particular it is, convex lower semicontinuous. But this function is given by:

$$V_{I_\Gamma}^B : H_m(\mathbb{C}) \to \mathbb{R} \cup \{+\infty\},\ C \to V_{I_\Gamma}^B(C) = \begin{cases} 0 \ if \ C \in {\lambda_B}^{-1}(\Gamma) \\ +\infty \ otherwise \end{cases} \quad (3.5)$$

In other words, it coincides with $I_{{\lambda_B}^{-1}(\Gamma)}$, the indicatrix function of ${\lambda_B}^{-1}(\Gamma)$. So the latter must itself be convex lower semicontinuous. As a consequence, ${\lambda_B}^{-1}(\Gamma)$is a convex closed (non empty) set of $H_m(\mathbb{C})$, indeed:

(i) ${\lambda_B}^{-1}(\Gamma)$ is convex because if $C, D \in {\lambda_B}^{-1}(\Gamma)$ and $t \in [0,1]$, we have by convexity of $I_{{\lambda_B}^{-1}(\Gamma)}$,

$$I_{{\lambda_B}^{-1}(\Gamma)}(tC + (1-t)D) \le tI_{{\lambda_B}^{-1}(\Gamma)}(C) + (1-t)\, I_{{\lambda_B}^{-1}(\Gamma)}(D) \qquad (3.6)$$

But $I_{{\lambda_B}^{-1}(\Gamma)}(C) = I_{{\lambda_B}^{-1}(\Gamma)}(D) = 0$ then necessarily $I_{{\lambda_B}^{-1}(\Gamma)}(tC + (1-t)D) = 0$ and $tC + (1-t)D \in {\lambda_B}^{-1}(\Gamma)$.

(ii) The set ${\lambda_B}^{-1}(\Gamma)$ is closed because $\left\{M \in H_m(\mathbb{C}) /\ +\infty \ge I_{{\lambda_B}^{-1}(\Gamma)}(M) > 0\right\} = H_m(\mathbb{C}) \setminus {\lambda_B}^{-1}(\Gamma)$ is an open set since $I_{{\lambda_B}^{-1}(\Gamma)}$ is lower semicontinuous.

4 Simplification of the proof of uniqueness of the solution of (E_k).

The Corollary 1.6 allows to notably simplify the proof of uniqueness of the solution of the equation (E_k) in comparison with [5].

Let φ_0 and φ_1 be two smooth k-admissible solutions of the equation (E_k) such that $\int_M \varphi_0\, \omega^m$=$\int_M \varphi_1\, \omega^m = 0$. For all $t \in [0,1]$, let us consider the function $\varphi_t = t\ \varphi_1 + (1-t)\,\varphi_0 = \varphi_0 + t\,\varphi$ with $\varphi = \varphi_1 - \varphi_0$. Let $P \in M$, and let us denote $h_k^P(t) = f_k\left(\left[\delta_i^j + g^{j\bar{l}}(P)\partial_{i\bar{l}}\varphi_t(P)\right]\right)$. We have $h_k^P(1) - h_k^P(0) = 0$ which is equivalent to $\int_0^1 {h_k^P}'(t)dt = 0$. But:

$${h_k^P}'(t) = \sum_{i,j=1}^{m}\left(\sum_{l=1}^{m}\frac{\partial f_k}{\partial B_i^l}\left(\left[\delta_i^j + g^{j\bar{l}}(P)\partial_{i\bar{l}}\varphi_t(P)\right]\right) g^{l\bar{j}}(P)\right)\partial_{i\bar{j}}\varphi(P)$$

Let us denote $\alpha_{ij}^t(P) = \sum_{l=1}^m \frac{\partial f_k}{\partial B_i^l}\left(\left[\delta_i^j + g^{j\bar{l}}(P)\partial_{i\bar{l}}\varphi_t(P)\right]\right) g^{l\bar{j}}(P)$. Therefore we obtain:

$$L\varphi(P) \coloneqq \sum_{i,j=1}^m a_{ij}(P)\ \partial_{i\bar{j}}\varphi(P) = 0 \qquad \text{with} \qquad a_{ij}(P) = \int_0^1 \alpha_{ij}^t(P)\ dt$$

We show easily that the matrix $\left[a_{ij}(P)\right]_{1\le i,j\le m}$ is hermitian [4, p. 53]. By the Corollary 1.6, we know that for all $t \in [0,1]$ and all points $m \in M$, $\lambda\left(g^{-1}\widetilde{g_{\varphi_t}}\right)(m) \in \Gamma_k$, namely that the functions $(\varphi_t)_{t\in[0,1]}$ are k-admissible.

We check then easily that the hermitian matrix $\left[a_{ij}(m)\right]_{1\le i,j\le m}$ is positive definite for all $m \in M$ [4, p. 54]. Consequently, the operator L is elliptic on M. But the map φ is C^∞ and satisfies $L\varphi = 0$, then by the Hopf maximum principle [10], we deduce that φ is constant on M. Besides $\int_M \varphi\, \omega^m = 0$, therefore we deduce that $\varphi \equiv 0$ on M namely that $\varphi_1 \equiv \varphi_0$ on M, which achieves the proof of uniqueness.

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The present results are an auxiliary, but independent, part of my PhD dissertation [4].

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